Solution to Exchanges 8.2 Puzzle: A Dutch Dutch Auction Clock Auction

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This is a solution to the editor's puzzle from issue 8.2 of SIGecom Exchanges. The puzzle is about finding a Bayesian equilibrium for a Dutch auction which can end according to a stochastic price schedule. The full puzzle [Conitzer 2009] can be found online at:
http://www.sigecom.org/exchanges/volume_8/2/puzzle.pdf.

Categories and Subject Descriptors: K.4.4 [**Computers and Society**]: Electronic Commerce

General Terms: Economics

Additional Key Words and Phrases: Dutch Auction, Stochastic Reserve Price

The puzzle asks us to find a Bayesian equilibrium for a Dutch auction with N bidders, where bidders' values are symmetrically and independently distributed on the interval $[0, 1]$.[1] Denote by $F(x)$ the cumulative distribution function according to which values are drawn, with f being the corresponding probability density function. The twist is that the object considered for sale is the Dutch auction clock itself, and it might break during the auction process. Let $W(p)$ describe the breaking probability function, i.e. the probability that the auction clock breaks after it reaches price p. If the clock breaks before the auction ends, its value drops to 0 and no bidder will buy it. The model, the distributions and the breaking schedule are common knowledge.

Interesting as it may be, the greatest virtue of the unstable Dutch auction clock story lies in its ability to provide us with intuitions for a much more widespread phenomenon, namely reserve prices. A discrete breaking schedule with only one value at which the clock breaks is completely equivalent to a known reserve price in a first-price sealed-bid auction. More general breaking schedules are equivalent to a hidden reserve price which is selected according to a known distribution. While it seems unlikely for a revenue maximizing auctioneer to use this type of mechanism (instead of the optimal auction), it is certainly plausible under mild deviations from the classic framework (e.g. [Li and Tan 2000]), and in fact hidden reserve prices are quite widespread in many real auction environments. Thus, the two approaches I present below apply equally to the original story, as well as to computing Bayesian equilibria when there is uncertainty about reserve prices.

[1] The author thanks Sergiu Hart, Déborah Marciano, Ilan Nehama, and Ran Shorrer for helpful discussions and suggestions.

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1. THE CASE OF $F(X) = X^M$ AND $W(P) = P^K$

Consider for a moment the case of $N = 2$, uniform values distribution $F(x) = x$, and a uniform breaking schedule $W(p) = p$. Assuming the existence of a symmetric and continuously differentiable increasing bidding function $\beta(v)$, we can write the utility of type v, when trying to bid as if she was of type r:

$$u(v, r) = (v - \beta(r))\, F(r) W(\beta(r)) = (v - \beta(r))\, r\beta(r) \tag{1}$$

In equilibrium the first-order condition when differentiating according to r should be zero, and therefore we get:

$$(v - \beta(v))\, (\beta(v) + v\beta'(v)) - \beta'(v) v\beta(v) = 0 \tag{2}$$

To solve this differential equation, suppose $\beta(v) = av$, and try to solve for a:

$$(v - av)(av + av) - a^2v^2 = 0 \Rightarrow av^2(2 - 3a) = 0 \Rightarrow \beta(v) = \frac{2}{3}v \tag{3}$$

It is easy to check that the second order condition holds as well. It is not surprising that the equilibrium bidding function is strictly above $\beta(v) = \frac{1}{2}v$ (the equilibrium bidding function without the breaking schedule), because each bidder wishes to raise her bid in the presence of the possibility to lose the auction not only to the competition, but also due to the stochastic breaking.

Furthermore, the above function should remind us of the symmetric equilibrium bidding function in a regular Dutch auction with $N = 3$ and $F(x) = x$. The reason the two cases are giving us the same bidding function is that we can treat the breaking schedule as if it is induced by actions of a non-strategic extra player, which bids uniformly on the unit interval. Since this type of bid can be thought of as multiplying the equilibrium strategy of a third player by a constant factor, the other players' maximization is the same as in the 3-person Dutch auction. This line of thought guides us in extending the solution for the slightly more general case of N bidders with uniform values, and breaking function of $W(p) = p^k$ (for any real number $k > 0$). We can treat the breaking schedule as if there were k more players (note that k can be non-integral). Specifically, we get the following symmetric Bayesian equilibria for N players:

$$W_1(p) = p \quad \Rightarrow \beta_1(v) = \left(\frac{N}{N+1}\right) v \tag{4}$$

$$W_2(p) = p^2 \quad \Rightarrow \beta_2(v) = \left(\frac{N+1}{N+2}\right) v \tag{5}$$

$$W_3(p) = \sqrt{p} \quad \Rightarrow \beta_3(v) = \left(\frac{N-\frac{1}{2}}{N+\frac{1}{2}}\right) v \tag{6}$$

Similarly, if the players' values are drawn according to $F(x) = x^m$, we can treat the breaking schedule $W(p) = p^k$ as if there were $\frac{k}{m}$ more bidders in the game. Technically, we have each bidder maximize over r:

$$u(v, r) = (v - \beta(r))\, F^{N-1}(r) W(\beta(r)) = (v - \beta(r))\, r^{m(N-1)} \beta^k(r) \tag{7}$$

And it is easy to verify that the equilibrium given by $\beta(v) = \left(\frac{m(N-1)+k}{m(N-1)+k+1}\right) v$ satisfies the first-order condition.

2. THE GENERAL CASE

In this section I assume, for the sake of mathematical convenience, that if the clock breaks on price p, then a bidder that suggested price p still wins the object at its original value. Imagine that $W(p)$ is given by a simple step function of the form:

$$W(p) = \begin{cases} 0 & \text{if } p \geq \hat{p} \\ 1 & \text{if } p < \hat{p} \end{cases} \tag{8}$$

for some $\hat{p} \in (0,1)$. In this case bidders with values below $\hat{p}$ can bid in equilibrium anything below $\hat{p}$. We can solve for the rest of the bidders in a similar manner to the way we usually solve, with the boundary condition $\beta(\hat{p}) = \hat{p}$. This is exactly parallel to the way we solve for first-price auction with a reserve price (see, for example, [Krishna 2002]).

The analysis becomes slightly trickier if we consider a breaking schedule made up of discrete jumps. Let $\{r_i\}_{i=0}^{R}$ be a division of the unit interval, i.e. $0 = r_0 < r_1 < \ldots < r_R = 1$, and let $\{q_i\}_{i=0}^{R-1}$ be positive numbers such that $\sum_{i=0}^{R} q_i \leq 1$. The discrete breaking schedule is given by:

$$W(p) = \begin{cases} 0 & \text{if } p = 1 \\ \sum_{i=j}^{R-1} q_i & \text{if } p \in [q_j, q_{j+1}) \end{cases} \tag{9}$$

In this case, we can construct an equilibrium using the following algorithm:

Algorithm 2.1. Initialize $\underline{t}_0 = 0$. For $k = 0 \ldots R-1$:
Step (k.1): Find the continuously increasing symmetric bidding function $\beta_k(v)$ for bidders with values $[\underline{t}_k, 1]$ (distributed according to $F|_{[\underline{t}_k,1]}$), with the boundary condition $\beta_k(\underline{t}_k) = r_k$.
Step (k.2): Within $[\underline{t}_k, 1]$, find the smallest value such that $(v - \beta_k(v))W(r_k) \leq (v - r_{k+1})W(r_{k+1})$. Set $\underline{t}_{k+1}$ to this value. Note that if $\beta_k(1) > r_{k+1}$ this must hold for some value (due to continuity). If no such value was found, set $\underline{t}_{k+1} = \underline{t}_k$, $\beta_{k+1} = \beta_k$, and skip directly to step (($k+1$).2).
The proposed equilibrium function is given by:

$$\beta(v) = \beta_{max\{l|v \geq t_l\}}(v) \tag{10}$$

LEMMA 2.2. *Algorithm 2.1 produces a symmetric Bayesian equilibrium for the Dutch auction with the discrete breaking schedule $W(p)$.*

PROOF. First note the algorithm outputs a monotonically increasing and piecewise continuous function, where each interval of continuity is a part of a symmetric Bayesian equilibrium found throughout the process. All intervals are of the form $[a, b)$, and all types which are interior to any of the intervals satisfy first-order conditions for being in equilibrium. Every bidder at the lower end of an interval is indifferent between bidding its current bid, or bidding the supremum of bids of players with lower values. Therefore she strictly prefers it to bidding as any of the bidder on the interval below it. It is both intuitive and easy to show that given three types $v_L < v_M < v_H$, incentive compatibility of type v_h with regard to v_M, and of type v_M with regard to v_L, leads to incentive compatibility of type v_H with regard to v_L. Use $P(b)$ as a shortcut for $\Pr(\text{win} \mid \text{bid} = b)$, and suppose:

$$(v_H - \beta(v_H))\, P(\beta(v_H)) \geq (v_H - \beta(v_M))\, P(\beta(v_M)) \tag{11}$$

$$(v_M - \beta(v_M))\, P(\beta(v_M)) \geq (v_M - \beta(v_L))\, P(\beta(v_L)) \tag{12}$$

then:

$$\begin{aligned}(v_H - \beta(v_L))\, P(\beta(v_L)) \leq (v_H - v_M)\, P(\beta(v_L)) + (v_M - \beta(v_M))\, P(\beta(v_M)) \leq \\ (v_H - v_M)\left(P(\beta(v_L)) - P(\beta(v_M))\right) + (v_H - \beta(v_H))\, P(\beta(v_H)) \leq \\ (v_H - \beta(v_H))\, P(\beta(v_H))\end{aligned} \tag{13}$$

And this gives us "downward" incentive compatibility. The same argument applies for "upward" incentive compatibility (using the minimality of the boundary types in the algorithm). The equilibrium condition is met because no bidder wants to bid outside Range(β). □

LEMMA 2.3. *Given individual value distribution $F(\cdot)$ and two breaking schedules $W_1(p)$ and $W_2(p)$ such that $\|W_1 - W_2\|_\infty < \epsilon$ for some $\epsilon > 0$, then any symmetric Bayesian equilibrium $\beta(\cdot)$ for the Dutch auction with breaking schedule $W_1(p)$ is a symmetric 2ϵ-Bayesian equilibrium for the Dutch auction with breaking schedule $W_2(p)$.*

PROOF. Suppose type v considers a deviation to play $b' \in supp(\beta)$ instead of $\beta(v)$ (deviating to $b' \notin supp(\beta)$ can always be improved by deviating to something in the support). Then we must have:

$$\begin{aligned}(v - \beta(v)) \cdot F^{n-1}(v) \cdot W_2(\beta(v)) \geq \\ (v - \beta(v)) \cdot F^{n-1}(v) \cdot W_1(\beta(v)) - (v - \beta(v)) \cdot F^{n-1}(v) \cdot \epsilon \geq \\ (v - b') \cdot F^{n-1}\left(\beta^{-1}(b')\right) \cdot W_1(b') - \epsilon \geq \\ (v - b') \cdot F^{n-1}\left(\beta^{-1}(b')\right) \cdot W_2(b') - (v - b') \cdot F^{n-1}\left(\beta^{-1}(b')\right) \cdot \epsilon - \epsilon \geq \\ (v - b') \cdot F^{n-1}\left(\beta^{-1}(b')\right) \cdot W_2(b') - 2\epsilon\end{aligned} \tag{14}$$

Showing that no bidders gain more than 2ϵ by deviating. □

Using both lemma 2.2 and lemma 2.3, we can find a symmetric ϵ-Bayesian equilibrium, for any $\epsilon > 0$, for a Dutch auction with an arbitrary breaking schedule $W(p)$ by taking a sufficiently close discrete approximation of $W(p)$, and then running algorithm 2.1.

PROPOSITION 2.4. *Assume that $W(p)$ is continuously differentiable on $[0, 1]$, and there is a sequence of symmetric ϵ_n-Bayesian equilibria, $\{\beta_{\epsilon_n}(\cdot)\}_{n=1}^{\infty}$ such that $\epsilon_n \to 0$ and $\beta_{\epsilon_n}(\cdot)$ pointwise converges to some $\beta(\cdot)$.*[2] *Then $\beta(\cdot)$ is a symmetric Bayesian equilibrium of the Dutch auction with breaking schedule $W(p)$ iff $\beta(\cdot)$ is strictly increasing.*

PROOF. Since $\beta(\cdot)$ is monotonically increasing, then not being strictly increasing implies that there is some interval on which $\beta(\cdot)$ is constant. But, this implies that the highest type on this interval would like to offer slightly more to gain a

[2] Helly's theorem [Brunk et al. 1956, Theorem 2] insures us that such a sequence indeed exists.

significant increase in her utility. As for the opposite direction, suppose $\beta(\cdot)$ is strictly increasing. Denote $\hat{W} = max_{x\in[0,1]} |W'(x)|$. For every type v and every possible deviation r, for every m select K_m such that for every $k > K_m$ we have $|\beta_{\epsilon_k}(v) - \beta(v)| < \frac{1}{m}$ and $|\beta_{\epsilon_k}(r) - \beta(r)| < \frac{1}{m}$, we must have then:

$$\begin{aligned}(v-\beta(v))\, F^{n-1}(v)W\left(\beta(v)\right) &\geq (v-\beta_{\epsilon_k}(v))\, F^{n-1}(v)W\left(\beta(v)\right) - \tfrac{1}{m} \geq \\ &(v-\beta_{\epsilon_k}(v))\, F^{n-1}(v)W\left(\beta_{\epsilon_k}(v)\right) - \tfrac{1+\hat{W}}{m} \geq \\ &(v-\beta_{\epsilon_k}(r))\, F^{n-1}(r)W\left(\beta_{\epsilon_k}(r)\right) - \tfrac{1+\hat{W}}{m} - \epsilon_k \geq \\ &(v-\beta(r))\, F^{n-1}(r)W\left(\beta_{\epsilon_k}(r)\right) - \tfrac{2+\hat{W}}{m} - \epsilon_k \geq \\ &(v-\beta(r))\, F^{n-1}(r)W\left(\beta(r)\right) - \tfrac{2+2\hat{W}}{m} - \epsilon_k\end{aligned} \tag{15}$$

As $\epsilon_k \to 0$ we get that the gain in utility by pretending to be type r is no more then $\frac{2+2\hat{W}}{m}$, and since m was arbitrarily picked we get that there is no incentive to deviate from $\beta(\cdot)$. There is no need to check other deviation (to bids which are not used in equilibrium by any type) since it is always better to bid as the highest type (due to continuity and no singletons in distributions). □

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