Competitive Equilibrium in Two Sided Matching Markets with General Utility Functions

SAEED ALAEI
University of Maryland, College Park
and
KAMAL JAIN
Mircosoft Research, Redmond
and
AZARAKHSH MALEKIAN
Northwestern University

Two sided matching markets are among the most studied models in market design. There is a vast literature on the structure of competitive equilibria in these markets, yet most of it is focused on quasilinear settings. General (non-quasilinear) utilities can, for instance, model smooth budget constraints as a special case. Due to the difficulty of dealing with arbitrary non-quasilinear utilities, most of the existing work on non-quasilinear utilities is limited to the special case of hard budget constraints in which the utility of each agent is quasilinear as long as her payment is within her budget limit and is negative infinity otherwise. Most of the work on competitive equilibria with hard budget constraints rely on some form of ascending auction. For general non-quasilinear utilities, such ascending auctions may not even converge in finite time. As such, almost all of the existing work on general non-quasilinear utilities have resorted to non-constructive proofs based on fixed point theorems or discretization. We present the first direct characterization of competitive equilibria in such markets. Our approach is constructive and solely based on induction. Our characterization reveals striking similarities between the payments at the lowest competitive equilibrium for general utilities and VCG payments for quasilinear utilities. We also show that lowest competitive equilibrium is group strategyproof for the agents on one side of the market (e.g., for buyers).

Categories and Subject Descriptors: G.0 [**Mathematics of Computing**]: General

General Terms: Economics, Theory, Algorithms

Additional Key Words and Phrases: Competitive Equilibrium, Non-quasilinear, Matching Market

A competitive equilibria (henceforth abbreviated to CE) in two sided matching markets with general utility functions is defined as follows. A CE is a bipartite matching along with monetary transfers between the matched parties, such that the outcome is stable in the sense that no two parties can form a matching pair and make monetary transfers in such a way that would improve both of their payoffs. It can be shown that these markets are equivalent to simpler markets in which there is a set of unit demand buyers and a set of heterogeneous goods in which the utility of each buyer is a joint function of the good she receives and the amount she pays,

Authors' addresses: `saeed@cs.umd.edu`, `kamalj@microsoft.com`,`a-malekian@northwetsern.edu`

but not necessarily a quasi-linear function of her payment.[1] In the buyers/goods market, a CE is an assignment of prices to goods together with a feasible allocation of goods to buyers such that every buyer receives her most preferred good at the announced prices and every unallocated good has a price of 0. Let $u_i^j(x)$ denote the utility of buyer i from receiving good j at price x. With quasi-linear utilities, this can be written as $u_i^j(x) = v_i^j - x$ in which v_i^j is the valuation of buyer i for good j. In this case utility is transferrable and therefore VCG is applicable. The efficient allocation can be computed using a maximum weight matching on the bipartite graph consisting of buyers/goods with the edge between buyer i and good j having a weight of v_i^j. The VCG payoffs/payments would then correspond to a minimum weighted cover on this graph [Leonard 1983]. For general (non-quasilinear) utilities, the functions $u_i^j(x)$ could be any continuous decreasing function of x. In this case, utility is non-transferrable and VCG is not applicable.

As a motivating example, consider a housing market in which each seller owns a house and each buyer wants to buy a house. Typically, buyers will have smooth budget constraints so their utility will not be quasilinear. For example, they may need to get loans/mortgages so the actual cost will include interests, fees, etc., in addition to the actual payment made to the seller.

In the abstract mathematical form, the problem we are looking at is a one-to-one matching with monetary transfers and general utilities as described by Demange and Gale[1985]. In this model, the set of CE corresponds exactly to the outcomes that are in the core. Demange and Gale also proved the lattice structure of the set of CE. The lattice structure was previously discovered by Shapley and Shubik [1971] for quasilinear utilities. The existence of CE for general utilities was proved by Quinzii [1984]. Quinzii showed that the game defined by this model is a *"Balanced Game"* and for general n-person balanced games it was previously shown by Scarf [1967] that the core is non-empty. Using a different method, Gale [1984] showed that for a more general class of preferences a CE always exists. Gale's proof is based on a generalization of the KKM lemma which is the continuous variant of the Sperner's lemma. Both of these proofs only show the existence of an equilibrium and are non-constructive. As such, they don't help much in understanding the properties of the equilibria.

In [Alaei et al. 2010], we give an exact inductive characterization of CE in two sided matching markets. First, observe that given either the prices of goods, or the payoffs of buyers at the equilibrium, we could easily construct the whole equilibrium prices/payoffs and allocations. Therefore, to fully characterize a CE we only need to specify either the price vector or the payoff vector at the equilibrium. Furthermore, the set of all CE form a complete lattice. In other words, taking the minimum(or maximum) of price vectors of two CE is itself a CE. Because of the lattice property, there exists a unique lowest and a unique highest CE (unique in terms of prices/payoffs). The following is an inductive characterization of lowest/highest CE.

(I) The payoff of an arbitrary buyer i at the lowest CE can be computed as follows. Remove buyer i from the market. Compute the prices of the highest CE of the

[1] Non-quasilinear utilities can be used for example to model smooth budget constraints.

market without i and offer these prices to buyer i. The payoff of buyer i from her most preferred good at these prices is equal to her payoff at the lowest CE of the whole market.

(II) The price of an arbitrary good j at the highest CE can be computed as follows. Remove good j from the market. Compute the payoffs at the lowest CE of the market without good j and ask every buyer to name a price for good j that would give them the same payoff (this could possibly be negative). The maximum of the named prices (or 0 if all named prices are negative) is equal to the price of good j at the highest CE of the whole market.

Observe that applying each one of the above rules reduces the size of the market by one, so by inductively applying the above two rules one can fully compute the prices/payoffs at the lowest/highest CE. Intuitively the above characterization can be interpreted as follows: The price of good j at the highest CE is the minimum price at which the market, as a whole, becomes indifferent between buying or not buying good j. In a sense, the price of good j at the highest CE determines how much good j is worth to the market as a whole. For buyer i to get good j, she has to pay the price of good j at the highest CE of the market without buyer i. In other words, the lowest price that buyer i has to pay to get good j is equal to how much good j is worth to the rest of the market. With a slight modification, the above rules can be used to compute any CE of the market. In particular, given lower bounds on prices/payoffs, one can construct a CE with prices/payoffs satisfying the lower bounds or determine that none exists (see [Alaei et al. 2010]).

It can be shown that the lowest CE is group strategyproof for buyers. In other words, there is no coalition of buyers who can misreport their utility functions in such a way that they all strictly improve without making side payments.

The characterization mentioned in the previous paragraph sheds new light on the structure of CE in two sided 1-to-1 matching markets. Furthermore, it possibly opens a new avenue for studying CE in n-to-1 or n-to-n matching markets. In markets with indivisibility, calculus is of little use. On the other hand, indivisibility allows the use of induction. Our characterization of CE in matching markets is an example of applying induction to such markets.

REFERENCES

Alaei, S., Jain, K., and Malekian, A. 2010. Competitive equilibrium in two sided matching markets with general utility functions. *arXiv* `http://arxiv.org/abs/1006.4696` .

Demange, G. and Gale, D. 1985. The strategy structure of two-sided matching markets. *Econometrica 53,* 4 (July), 873–88.

Gale, D. 1984. Equilibrium in a discrete exchange economy with money. *International Journal of Game Theory 13,* 1, 61–64.

Leonard, H. B. 1983. Elicitation of honest preferences for the assignment of individuals to positions. *Journal of Political Economy 91,* 3, 461–79.

Quinzii, M. 1984. Core and competitive equilibria with indivisibilities. *International Journal of Game Theory 13*, 41–60.

Scarf, H. E. 1967. The core of an n person game. *Econometrica 35,* 1 (January).

Shapley, L. S. and Shubik, M. 1971. The assignment game I: The core. *International Journal of Game Theory 1*, 111–130. 10.1007/BF01753437.