Solution to Exchanges 9.1 Puzzle: Borrowing as Cheaply as Possible

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This is a solution to the Editor's Puzzle published in Issue 9.1 of SIGecom Exchanges [Conitzer 2010]. The puzzle is about finding the least expensive way for a player to borrow a certain amount of money from others under some given constraints and can be found at
`http://www.sigecom.org/exchanges/volume_9/1/puzzle.pdf`.

1. THE MODEL

This problem can be easily modeled as an instance of the *minimum cost flow* problem. In this problem, we are given a directed graph $G = (V, E)$ (or network), where each node is associated with a demand value, and each arc is characterized by two non-negative values denoting its cost and capacity. The goal is to regulate the flow of a single commodity in the network in a way that satisfies all demands, minimizes the cost, and obeys the capacity constraints of each edge.

The instance is depicted in Figure 1(a). Each person is represented by a node. Specifically, nodes A, B, D, E, G, H and Z correspond to Alice, Bob, Denise, Ed, Grace, Harry, and Zoe, respectively. The commodity we want to transfer through the network is money. The demand d_v of node $v \in \{\text{A, B, D, E, G, H, Z}\}$ denotes the amount of money the corresponding person originally has (if $d_v < 0$), or wants to receive (if $d_v > 0$); nodes for which $d_v = 0$ have no demand label in the figure. There is a directed arc (v, w) from node v to node w if v is willing to lend to w, and it is associated with two values: a cost c_{vw} denoting the profit the former wants to gain by lending \$100 to the latter, and a capacity u_{vw} denoting the maximum amount of money v is willing to lend to w. Note that the demands and capacities are expressed in \$100 bills in the figure.

2. THE SOLUTION

We use a criterion for the optimality of a solution based on *residual networks*. First, we show how to construct such a network based on a given flow. The residual network has the same node set as the original one. Regarding its arc set, assume every directed arc (v, w) in E is replaced by two directed arcs: the arc (v, w) has cost c_{vw} and residual capacity $r_{vw} = u_{vw} - x_{vw}$, and the arc (w, v) has cost $-c_{vw}$ and residual capacity $r_{wv} = x_{vw}$, where x_{vw} is the current flow from v to w. Intuitively, we can send back the flow sent from v to w and decrease the total cost. The arc set of the residual network consists of a subset of these arcs, namely the ones with

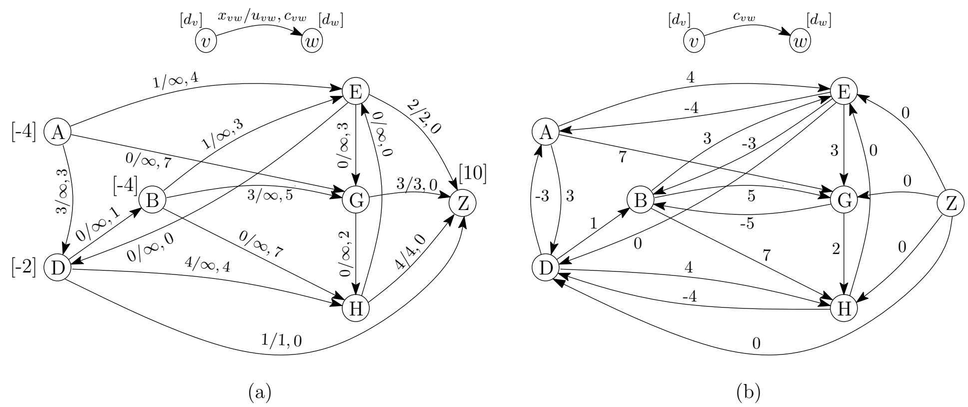

Fig. 1. (a) The original instance of the problem. x_{vw} denotes the flow from node v to node w in the optimal solution. (b) The residual network given the optimal flow. Capacities are left out for conciseness, and only arcs with positive capacities are shown.

strictly positive residual capacity. If the residual network has a directed cycle of negative total cost, that means we can decrease the cost sending flow through that cycle. Otherwise, the corresponding flow is optimal with respect to its cost. The "Negative Cycle Optimality Condition" is proved in [Busacker and Saaty 1965].

Let x be a feasible solution for this network flow problem, i.e., which satisfies all demands and obeys the capacities. x is shown in Figure 1(a) and is also stated in terms of the original problem as follows:

> Alice lends \$100 to Ed (cost 4), and her remaining \$300 to Denise (cost 9), which Denise lends to Harry together with \$100 of her own (cost 16). Harry now has \$400 that he lends to Zoe (cost 0). Denise lends her remaining \$100 to Zoe (cost 0). Also, Bob lends \$100 to Ed (cost 3) and his remaining \$300 to Grace (cost 15), which Grace in turn lends to Zoe (cost 0). Finally, Ed lends \$200 to Zoe (cost 0).

The residual network based on x is depicted in Figure 1(b). We can see that there are no negative cost directed cycles in the residual graph, which implies that x is a minimum cost feasible flow. The total cost of x is \$47, hence, it will cost Zoe \$47 to borrow \$1000 from her friends.

Min cost flow problems also have another nice property, known as the Integrality Property, stating that "if all edge capacities and demands of the nodes are integers, then the minimum cost flow problem always has an integer solution (if it has a solution)". This property implies that problems similar to the one we examined, i.e., which can be modeled as minimum cost flow problems, always have solutions involving only exchanges of multiples of 100 as long as the input numbers are multiples of \$100 as well, which answers the final question of the puzzle.

REFERENCES

Busacker, R. G. and Saaty, T. L. 1965. *Finite Graphs and Networks: An Introduction with Applications.* McGraw-Hill, New York, NY.

Conitzer, V. 2010. Editor's puzzle: Borrowing as cheaply as possible. *SIGecom Exchanges 9.1.*