

The Price of Anarchy in Games of Incomplete Information

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We outline a recently developed theory for bounding the inefficiency of equilibria in games of incomplete information, with applications to auctions and routing games.

Categories and Subject Descriptors: F.0 [**Theory of Computation**]: General

General Terms: Algorithms, Economics, Theory

Additional Key Words and Phrases: Price of anarchy, incomplete information, congestion games, auctions

1. INTRODUCTION

Every student of game theory learns early and often that *equilibria are inefficient*. Such inefficiency is ubiquitous, and is present in many real-world situations and for many different reasons: in Prisoner’s Dilemma-type scenarios; from uninternalized negative externalities in the tragedy of the commons and in games with congestion effects; from uninternalized positive externalities with a public good or with network effects; from a failure to coordinate in team games; and so on.

The past ten years have provided an encouraging counterpoint to this widespread equilibrium inefficiency: in a number of interesting application domains, game-theoretic equilibria *provably approximate* the optimal outcome. Phrased in modern jargon, the *price of anarchy* — the worst-case ratio between the objective function value of an equilibrium, and of optimal outcome — is close to 1 in many interesting games.

Essentially all initial work on the price of anarchy studied *full-information* games, where all players’ payoffs are assumed to be common knowledge. Now that the study of equilibrium inefficiency has grown in scope and considers strategically interesting auctions and mechanisms, there is presently a well-motivated focus on the price of anarchy in games of *incomplete information*, where players are uncertain about each others’ payoffs. A recent paper [Roughgarden 2012], outlined below, develops tools for bounding the price of anarchy in such games.

2. EXECUTIVE SUMMARY: PRICE OF ANARCHY BOUNDS FOR BAYES-NASH EQUILIBRIA VIA EXTENSION THEOREMS (I.E., WITHOUT THE PAIN)

Pure-strategy Nash equilibria — where each player deterministically picks a single action — are often easy to reason about. For this reason, the price of anarchy of

This work is supported in part by NSF grant CCF-1016885, an ONR PECASE Award, and an AFOSR MURI grant.

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a game is often analyzed, at least initially, only for the game’s pure-strategy Nash equilibria. But as much as he or she might want to, the conscientious researcher cannot stop there. Performance guarantees for more general classes of equilibria are crucial for several reasons (for example, pure-strategy Nash equilibria need not exist). In particular, a fundamental assumption behind the Nash equilibrium concept is that all players’ preferences are common knowledge, and this assumption is violated in most auction and mechanism design contexts, where participants have private information.

Extending price of anarchy bounds beyond pure Nash equilibria is an extremely well motivated activity, but it is also potentially dispiriting, for two reasons. The first is that the analysis generally becomes more complex, with one or more unruly probability distributions obfuscating the core argument. The second is that enlarging the set of permissible equilibria can only degrade the price of anarchy (which is a worst-case measure). Thus the work can be difficult, and the news can only be bad.

Can we obtain price of anarchy bounds for more general equilibrium concepts without working any harder than we already do to analyze pure-strategy Nash equilibria? Ideal would be an *extension theorem* that could be used in the following “black-box” way: (1) prove a bound on the price of anarchy of pure-strategy Nash equilibria of a game; (2) invoke the extension theorem to conclude immediately that the exact same approximation bound applies to some more general equilibrium concept. Such an extension theorem would dodge both potential problems with generalizing price of anarchy bounds beyond pure Nash equilibria — no extra work, and no loss in the approximation guarantee.

Since there are plenty of games in which (say) the worst mixed-strategy Nash equilibrium is worse than the worst pure-strategy Nash equilibrium (like “Chicken”), there is no universally applicable extension theorem of the above type. The next-best thing would be an extension theorem that applies under some conditions — perhaps on the game, or perhaps *on the method of proof used* to bound the price of anarchy of pure Nash equilibria. If such an extension theorem existed, it would reduce proving price of anarchy bounds for general equilibrium concepts to proving such bounds *in a prescribed way* for pure-strategy Nash equilibria.

The first example of such an extension theorem was given in [Roughgarden 2009], for full-information games. The key concept in [Roughgarden 2009] is that of a *smooth* game. Conceptually, a full-information game is smooth if the objective function value of every pure-strategy Nash equilibrium \mathbf{a} can be bounded using the following minimal recipe:

- (1) Let \mathbf{a}^* denote the optimal outcome of the game.
- (2) Invoke the Nash equilibrium hypothesis once per player, to derive that each player i ’s payoff in the Nash equilibrium \mathbf{a} is at least as high as if it played a_i^* instead. *Do not use the Nash equilibrium hypothesis again in the rest of the proof.*
- (3) Use the inequalities of the previous step, possibly in conjunction with other properties of the game’s payoffs, to prove that the objective function value of \mathbf{a} is at least some fraction of that of \mathbf{a}^* .

Many interesting price of anarchy bounds follow from “smoothness proofs” of this

type. The main extension theorem in Roughgarden [2009] is that every price of anarchy bound proved in this way — seemingly only for pure Nash equilibria — automatically extends to every mixed-strategy Nash equilibrium, correlated equilibrium, and coarse correlated equilibrium of the game.

Our new paper [Roughgarden 2012] presents a general extension theorem for games of incomplete information, where players’ private preferences are drawn independently from prior distributions that are common knowledge. This extension theorem reduces, in a “black-box” fashion, the task of proving price of anarchy bounds for mixed-strategy Bayes-Nash equilibria to that of proving such bounds in a prescribed way for pure-strategy Nash equilibria in every induced game of full information (after conditioning on all players’ preferences). With this extension theorem, one can prove equilibrium guarantees for games of incomplete information without ever leaving the safe confines of full-information games.

We conclude this section with an overview of the main points of the paper [Roughgarden 2012].¹

- (1) Smooth games of incomplete information are defined. The definition is slightly stronger, in a subtle but important way, than requiring that every induced full-information game is smooth.
- (2) There is an extension theorem for smooth games of incomplete information: price of anarchy bounds for pure Nash equilibria for all induced full-information games extend automatically to all mixed-strategy Bayes-Nash equilibria with respect to a product prior distribution over players’ preferences.
- (3) Many games of incomplete information for which the price of anarchy has been studied are smooth in this sense. Thus this extension theorem unifies much of the known work on the price of anarchy in games of incomplete information.
- (4) This extension theorem implies new bounds on the price of anarchy of Bayes-Nash equilibria in congestion games with incomplete information.
- (5) For Bayes-Nash equilibria in games with correlated player preferences, there is no general extension theorem for smooth games. (Additional conditions under which the extension can be recovered are given in Caragiannis et al. [2012].)

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¹Some of our results were also obtained, subsequently but independently, in [Syrgkanis 2012]. There are also results in [Syrgkanis 2012] for mechanisms with first-price payment rules, which are not considered in [Roughgarden 2012].