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Editor’s Introduction

Yiling Chen
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It is my pleasure to introduce Issue 11.1 of SIGecom Exchanges. This issue features seven research letters on a broad spectrum of topics and a solution to the ‘Borrowing in the Limit as Our Nerdiness Goes to Infinity’ puzzle of Issue 10.2. (And see below for a bounty on the most recent puzzle.)

In the first letter, Roth reviews recent progress at the boundary of mechanism design and differential privacy. Focusing on the sensitive surveyor’s problem—a surveyor designs a mechanism to obtain an accurate estimate of some population statistic when agents in the population care about privacy—he discusses results on designing direct revelation mechanisms and take-it-or-leave-it mechanisms to achieve non-trivial accuracy.

The second letter is about rationing problems, a type of fair division problem. A standard rationing problem allocates a divisible resource to a set of agents, each demanding some amount of the resource with total demand exceeding the available resource. Moulin and Sethuraman revisit the theory of standard rationing problems, in particular the consistency requirement, in a bipartite setting where there are multiple substitutable resources, and each agent has a global demand for resources but has access to only a subset of them.

Roughgarden outlines in the third letter an exciting, recent theory for bounding the price of anarchy (PoA) in games of incomplete information. In his earlier work, focusing on games with complete information, Roughgarden develops an extension theorem, via the concept of smooth games, that automatically extends the PoA bounds of pure-strategy Nash equilibria to some more general equilibrium concepts including mixed-strategy Nash equilibria. The recent theory defines smooth games of incomplete information and establishes the PoA bounds of mixed-strategy Bayes-Nash equilibria by directly extending the PoA bounds of pure-strategy Nash equilibria in induced games of complete information.

In the next letter, Nikolova and Stier-Moses introduce uncertainty and risk-aversion into routing games. In a model of stochastic edge delay and risk-averse players, they investigate equilibrium existence, characterization, and PoA in routing games and suggest open questions.

The fifth letter by Caragiannis et al. considers computing $\rho$-approximate Nash equilibria, where no unilateral deviation can improve a player’s payoff by a factor larger than $\rho$. For weighted and unweighted congestion games with polynomial latency functions of constant maximum degree, Caragiannis et al. describe positive algorithmic results for computing $O(1)$-approximate pure-strategy Nash equilibria.

In the sixth letter, Haghpanah et al. discuss their work on approximating revenue-maximizing auctions in single parameter settings. The optimal Myerson auction requires solving an NP-hard optimization problem with virtual values of bidders. Instead of approximating the optimal solution of this problem point-wise (i.e., for...
each realization of values) as in prior work, Haghpanah et al. approximate the optimal solution on average, with respect to the distribution of virtual values.

The last letter by Bagchi et al. presents the allocation and trading of carbon dioxide emission credits as mechanism design problems. The allocation of carbon credits to organizations and the subsequent buying and selling of carbon credits among organizations call for well designed mechanisms to achieve optimal emission reduction. Bagchi et al. describe some initial results on the allocation problem.

Our puzzle editor, Daniel Reeves, brings us Shorrer’s solution to the Borrowing in the Limit as Our Nerdiness Goes to Infinity puzzle. Armed with the definition of the time-value of money – i.e., what precisely we mean by an interest rate – the most elegant solution involves a simple integral, summing up the stream of infinitesimal payments. Shorrer provides two variants of the solution and discusses potential practical implications.

As we haven’t received a correct solution to the Contingency Exigency puzzle of Issue 10.3, there is no new puzzle in this issue. Believing in incentives, randomness, and general nerdery, our puzzle editor has put aside up to $500 as a bounty, to help ensure we get one. Here are the crazy details:

The best – fastest and most elegant – solutions to Contingency Exigency will share a bounty. The bounty amount is a $\mathcal{U}[0, 500]$ random variable. Of course you won’t want to trust me to instantiate the random variable so we’ll use xkcd’s geohashing algorithm: http://xkcd.com/426/. The bounty amount is $500$ times Greenwich’s longitude value on the date of submission of the first correct solution. (Note that you could strategically delay your submission to, in expectation, increase the bounty, at risk of getting scooped.) As to how the bounty is shared between the fastest and most elegant, that’s entirely to my discretion, but resubmissions are allowed, so you can aim to be both and make the question of sharing moot.

Finally, I would like to thank our Information Director, Felix Fischer, who as always has been very helpful in putting the issue together.
Buying Private Data at Auction: The Sensitive Surveyor’s Problem

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In this letter, we survey some recent work on what we call the *sensitive surveyor’s problem*. A curious data analyst wishes to survey a population to obtain an accurate estimate of a simple population statistic: for example, the fraction of the population testing positive for syphilis. However, because this is a statistic over sensitive data, individuals experience a cost for participating in the survey as a function of their loss in privacy. Agents must be compensated for this cost, and moreover, are strategic agents and will mis-report their cost if doing so is beneficial for them. The goal of the surveyor is to manage the inevitable tradeoff between the cost of the survey, and the accuracy of its results.

Categories and Subject Descriptors: J.4 [Social and Behavioral Sciences]: Economics

General Terms: Privacy, Mechanism Design

1. INTRODUCTION

Consider the following stylized problem of the sensitive surveyor Alice. She is tasked with conducting a survey of a set of \( n \) individuals \( N \), to determine what proportion of the individuals \( i \in N \) satisfy some property \( P(i) \). Her ultimate goal is to discover the true value of this statistic, \( s = \frac{1}{n} \cdot |\{i \in N : P(i)\}| \), but if that is not possible, she will be satisfied with some estimate \( \hat{s} \) such that the error, \( |\hat{s} - s| \), is minimized. We will adopt a notion of accuracy based on large deviation bounds, and say that a surveying mechanism is \( \alpha \)-accurate if \( \Pr[|\hat{s} - s| \geq \alpha] \leq \frac{1}{3} \). The inevitable catch is that individuals will not participate in the survey for free. \( P(i) \) may be an embarrassing or otherwise sensitive predicate (e.g. it may represent the presence of a disease, a juvenile taste in movies, a political stance, or any other property that might cause its owner to come to grief if it were to be revealed). Therefore, individuals will experience some cost as a function of their loss in privacy when they interact with Alice, and will insist on being compensated for this loss. To make matters worse, these individuals are rational (i.e. selfish) agents, and are apt to mis-report their costs to Alice if doing so will result in a financial gain. This places the sensitive surveyor’s problem squarely in the domain of mechanism design, and requires Alice to develop a scheme for trading off statistical accuracy with cost, all while managing the incentives of the individuals in \( N \).

Before we go on to describe the recent work in this area, we remark that this stylized problem is not only relevant to a surveyor, but to any organization that

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1It is the data that is sensitive, not necessarily the surveyor herself. As we will see, Alice may or may not be sensitive.

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makes use of collections of potentially sensitive data to provide some service, or to otherwise extract value from it. This includes, for example, the use of search logs to provide search query completion and the use of browsing history to improve search engine ranking, the use of social network data to select display ads and to recommend new links, and the myriad other data-driven services now available on the web. In all of these cases, value is being derived from the statistical properties of collections of sensitive data in exchange for some payment\(^2\).

Collecting data in exchange for some fixed price will inevitably lead to a biased estimate of population statistics, because such a scheme will result in collecting data only from those individuals who value their privacy less than the price being offered. To obtain an accurate estimate of the statistic, it is therefore natural to consider buying private data using an auction, which was recently considered by Ghosh and Roth [2011]. There are two obvious obstacles which one must confront when conducting an auction for private data, and one additional obstacle which is less obvious but more insidious. The first obstacle is that one must have a quantitative formalization of “privacy” which can be used to measure agents’ costs under various operations on their data. We discuss this in the next section: the short answer is that we are fortunate that the internet-scale use of private data has dovetailed with the development of the notion of differential privacy, which is an excellent quantitative formalization for how one may trade off privacy for utility. The second obstacle is that the objective that we wish to trade off with cost is statistical accuracy, which is distinct from the objectives commonly studied in mechanism design.

The final, more insidious obstacle, is that we expect that an individual’s cost for privacy loss should be highly correlated with his private data itself! Suppose, for example, that Bob reports a low value for privacy, but after being observed to visit an oncologist, Bob revises his value for privacy to be much higher. Although we only know Bob’s value for privacy, and have not explicitly been shown his medical records, this is disclosive because Bob’s cancer status is likely correlated with his value for privacy. More to the point, suppose that in the first step of a survey of cancer prevalence, we ask each individual to report their value for privacy, with the intention of then running an auction to choose which individuals to buy data from. If agents report truthfully, we may find that the reported values naturally form two clusters: low value agents, and high value agents. In this case, we may have learned something about the population statistic even before collecting any data or making any payments – and therefore, the agents will have already experienced a cost. As a result, the agents may not be incentivized to report their true values, and this could again serve to introduce a bias in the survey results. This phenomenon makes direct revelation mechanisms problematic in auctions for private data, and it is what most distinguishes this problem from classical mechanism design.

2. DIFFERENTIAL PRIVACY: QUANTIFYING PRIVACY’S UTILITY

“Differential Privacy” takes the position that privacy is a property of a process, and not a property of a piece of information. It asserts that people should care about
the privacy of their data to the extent that the use of that data causes additional harm to befall them, compared to if their data was not used at all. The following definition of differential privacy is syntactically different from the one originally given in [Dwork et al. 2006; Dwork 2006], but is easily seen to be equivalent, and is particularly well suited to its use in mechanism design. Let us suppose that each individual has a piece of private data drawn from some abstract domain \( X \). A pair of neighboring databases \( D, D' \in X^n \) are said to be neighboring if they differ in at most one coordinate (i.e. if they differ only in the data of a single individual). An algorithm is then a randomized mapping \( M : X^n \rightarrow T \) to some abstract range \( T \) of outcomes.

**Definition 2.1.** An algorithm \( M : X^n \rightarrow T \) is \( \epsilon \)-differentially private if for all pairs of neighboring databases \( D \) and \( D' \), and for all utility functions \( u : T \rightarrow \mathbb{R} \):

\[
\exp(-\epsilon) \mathbb{E}[u(M(D))] \leq \mathbb{E}[u(M(D'))] \leq \exp(\epsilon) \mathbb{E}[u(M(D'))]
\]

\( \epsilon \) is typically taken to be some value \( < 1 \), and so \( \exp(-\epsilon), \exp(\epsilon) \) should be thought of as factors of \((1 - \epsilon)\) and \((1 + \epsilon)\) respectively. In words, what a promise of differential privacy guarantees is that simultaneously for all agents, no matter what utility function they may have over the outcome of the mechanism, participation in a computation \( M \) cannot negatively (or positively) change that expected utility by more than a \((1 + \epsilon)\) factor.

This naturally motivates a way to measure the cost to an agent of allowing their private data to be used by some mechanism \( M \): If \( D \) is the database that includes agent \( i \)'s data, and \( D' \) is the identical database with agent \( i \)'s data removed, then the cost to agent \( i \) of participating is \( c_i = \mathbb{E}[u(M(D))] - \mathbb{E}[u(M(D'))] \). If \( M \) is differentially private, then this cost is guaranteed to be bounded by \( \epsilon \mathbb{E}[u(M(D'))] \). This motivates a particularly simple linear form of privacy cost in terms of the differential privacy guarantee \( \epsilon \) with which his data is protected: \( c_i(\epsilon) = v_i \epsilon \), where \( v_i \) is a privately known value parameter.

3. **DIRECT REVELATION MECHANISMS**

Armed with a means of quantifying an agent \( i \)'s loss for allowing his data to be used by an \( \epsilon \)-differentially-private algorithm \( (c_i(\epsilon) = \epsilon \cdot v_i) \), we are almost ready to describe results for the sensitive surveyor’s problem. It remains to define what exactly the data domain \( X \) is. We will consider two models. In both models, we will associate with each individual a bit \( b_i \in \{0, 1\} \) which represents whether they satisfy the sensitive predicate \( P(i) \), as well as a value for privacy \( v_i \in \mathbb{R}^+ \).

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3McSherry and Talwar [2007] were the first to observe that differential privacy implied this alternative formulation and used it in the context of mechanism design. I first saw this formulation presented as the definition of differential privacy in a talk given by Kobbi Nissim on his work in privacy and mechanism design [Nissim et al. 2012; Nissim, Orlandi, and Smorodinsky 2012].

4Although this form of utility is simple and is what is used in Ghosh and Roth [2011], it can be problematic. Indeed, the models in [Ligett and Roth 2012; Fleischer and Lyu 2012; Nissim, Orlandi, and Smorodinsky 2012] use slightly modified models of privacy cost, and those in [Xiao 2011; Chen et al. 2011] use a significantly different measure. The “right” measure for privacy utility is still up for debate: see [Nissim, Orlandi, and Smorodinsky 2012; Ligett and Roth 2012] for further discussion on this issue.
In the insensitive value model, we calculate the $\epsilon$ parameter of the private mechanism by letting its domain be $X^n = \{0,1\}^n$: i.e. we measure privacy cost only with respect to how the mechanism treats the sensitive bit $b_i$, and ignore how it treats the reported values for privacy, $v_i$.

In the sensitive value model, we calculate the $\epsilon$ parameter of the private mechanism by letting its domain be $X^n = (\{0,1\} \times \mathbb{R}^+)^n$: i.e. we measure privacy cost with respect to how it treats the pair $(b_i, v_i)$ of bit/value pairs for each individual.

Intuitively, the insensitive value model treats individuals as ignoring the potential privacy loss due to correlations between their values for privacy and their private bits, whereas the sensitive value model treats individuals as assuming these correlations are worst-case, and that their values $v_i$ are just as disclosive as their private bits $b_i$. The main results of Ghosh and Roth [2011] are that in the insensitive value model, it is possible to derive approximately optimal direct revelation mechanisms that achieve high accuracy and low cost. On the other hand, in the sensitive value model, no individually rational direct revelation mechanism can achieve any non-trivial accuracy.

Note that here we are considering a setting in which private data and costs are adversarially chosen. If we are willing to assume a known prior on agent costs (but still assume adversarially chosen private bits $b_i$), then it is possible to improve on the results of Ghosh and Roth [2011], and derive Bayesian optimal mechanisms for the sensitive survey problem. This is considered in Roth and Schoenebeck [2012], but we will not have space to discuss these results in the present note.

4. TAKE IT OR LEAVE IT MECHANISMS

Given the impossibility result proven in Ghosh and Roth [2011] for the sensitive value model, the immediate question is how we may circumvent it. In recent work, two methods have been proposed, which we briefly summarize here. Both approaches abandon direct revelation mechanisms in favor of mechanisms which offer individuals binary take-it-or-leave-it offers, but both also require subtle changes in how individuals are modeled as valuing privacy. We will not have space to explain these subtleties here, but readers are directed to the papers Fleischer and Lyu [2012; Ligett and Roth [2012] for more details.

4.1 Circumventing Impossibility with a Sensitive Surveyor

Suppose an individual is approached with a take it or leave it offer: “If you let us use your bit $b_i$ in an $\epsilon$-differentially private manner, I will give you $10.” An individual might be reluctant to respond to such an offer, because the very act of responding might reveal whether his value $v_i$ is such that $v_i \geq 10/\epsilon$ or not, and if values are correlated with private data, this might reveal something about his bit $b_i$ beyond that which is revealed through the differentially private computation. To model such correlations, Fleischer and Lyu [2012] assume that each individual’s value $v_i$ is drawn independently from one of two known priors: $v_i \sim F_0$ if $b_i = 0$, and $v_i \sim F_1$ if $b_i = 1$. Alice, the surveyor, knows both priors, but does not know whether $b_i = 0$ or $b_i = 1$. Using this assumption, Fleischer and Lyu use an elegant idea which allows Alice to make a take-it-or-leave-it offer which an agent can truthfully decide to
accept or reject without revealing anything about his private bit! The idea is this: Alice may choose some acceptance probability $q \in [0, 1]$. Although she does not know the bit of the agent she is surveying, she can pick two values $p_0, p_1$ such that $\Pr_{v \sim F_0} [v \leq p_0 / \epsilon] = q$ and $\Pr_{v \sim F_1} [v \leq p_1 / \epsilon] = q$. Alice can then offer the following take-it-or-leave-it offer to each agent: “If you accept the offer and your (verifiable) bit is 0, I will pay you $p_0$ dollars. If you accept the offer and your bit is 1 I will pay you $p_1$ dollars.” The beauty of this solution is that no matter what private bit the agent has, he will accept the offer with probability $q$ (where the probability is taken over the draw of his value from the corresponding prior) and reject the offer with probability $(1 - q)$. Therefore, nothing can be learned about his private bit from his participation decision, and so he has no incentive not to respond to the offer truthfully. Using this idea, Fleischer and Lyu [2012] develop approximately optimal truthful take-it-or-leave-it mechanisms that can be used whenever exact priors $F_0$ and $F_1$ are known. The solution is to make Alice’s query to each agent more sensitive to their privacy concerns so that their participation decisions do not reveal their private data.

4.2 Circumventing Impossibility with an Insensitive Surveyor

What if agent costs are again determined adversarially, and there are no known priors? Ligett and Roth [2012] give an alternative solution for this case, again based on making take-it-or-leave-it offers. To circumvent the impossibility result of Ghosh and Roth [2011], Alice is here granted with one additional power: the ability to accost random members of the population on the street, and present them with a take-it-or-leave-it offer. Once individuals are presented with an offer, they are free to accept it or refuse it however they see fit. But they may not choose to have never even heard the offer, and if they reject the offer (perhaps just by walking away), their non-participation decision is observed by Alice. This can be seen as a weakening of the individual rationality condition: because costs may be correlated with private data, merely by rejecting an offer and walking away, Alice may learn something about the surveyed individual. If the individual did not accept the offer, he receives no payment, and yet still experiences some cost! This ends up giving a semi-truthfulness guarantee. Whenever Alice makes an offer of $p$ dollars in exchange for $\epsilon$-differential privacy, a rational agent will accept whenever $p \geq ev_i$. On the other hand, rational agents may or may not accept offers that are below their cost – because they will still experience some cost by walking away. But these deviations away from “truthfulness” are in only one direction, and only help Alice, whose aim it is to compute an accurate population statistic, and does not necessarily care about protecting privacy for its own sake. Here, Ligett and Roth [2012] are able to again obtain non-trivial accuracy (circumventing the impossibility result of Ghosh and Roth [2011]) even in the sensitive value model by making Alice insensitive to the privacy concerns of the agents she surveys, by making offers that they can refuse (but can’t avoid).

5. OTHER RELATED WORK

This note has discussed only the sensitive surveyors problem, but it should be noted that there is a significant amount of work on other problems at the intersection of privacy and mechanism design. Differential privacy was first used as a tool
in mechanism design (and proposed as a solution concept) by McSherry and Talwar [2007]. Gupta et al. [2010] also used differential privacy as a solution concept. These results produced only approximately truthful mechanisms; recently Nissim et al. [2012] showed how to combine differentially private mechanisms with “imposition mechanisms” to derive new mechanisms which are exactly truthful and do not require payments. In recent insightful work, Xiao [2011] studied the problem of mechanism design for agents who explicitly value privacy, and asked whether mechanisms of the sort studied in Nissim et al. [2012] could be made truthful even in the presence of such agents. Recently, Nissim, Orlandi, and Smorodinsky [2012] and Chen et al. [2011] gave positive answers to this question, each in a slightly different model.

REFERENCES


Rationing Problems in Bipartite Networks

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The standard theory of rationing problems is extended to the bipartite context. The focus is on consistency, a compelling rationality property of fair division methods in the standard setting.

Categories and Subject Descriptors: G.2.3 [Mathematics of Computing]: Discrete Mathematics—Applications
General Terms: Algorithms; Economics; Theory
Additional Key Words and Phrases: Games, Networks/Graphs, Fair Division, Consistency

We start with a brief description of standard rationing problems, followed by an overview of the main results of our paper [Moulin and Sethuraman 2011]. The final sections report on more general results (work in progress) and some directions for further research.

1. STANDARD RATIONING PROBLEMS

Standard rationing problems are the simplest type of fair division problems, where an amount $r$ of some divisible resource (e.g., money) must be divided between the agents $i$ of a given set $N$, who each have a claim (or demand) $x_i$ on the resources, and the total claim exceeds the available resources: $\sum_N x_i > r$. Bankruptcy and the distribution of emergency supplies are typical examples.

A rationing method $h$ selects for every $N$, every vector $x$ of claims, and every non-negative $r$ such that $r \leq x_N$ (with the standard notation $x_N = \sum_N x_i$), a vector of shares $y = h(N, x, r)$ such that $0 \leq y \leq x$ and $y_N = r$. Three benchmark methods emerge clearly from the abundant axiomatic (see the surveys [Moulin 2002] and [Thomson 2003]), as well as the experimental and social psychology [Cook and Hejtvedt 1983; Deutsch 1975] literatures. The proportional method allocates shares in proportion to claims, so agent $i$’s share is $y_i = (x_i/x_N)r$. The uniform gains (a.k.a. equal awards) method equalizes shares as much as permitted by the constraint $y_i \leq x_i$: therefore $y_i = \min\{\lambda, x_i\}$, where $\lambda$ is chosen so that $\sum_N y_i = r$. Finally the uniform losses (a.k.a. equal losses) method equalizes losses (i.e., the differences $x_i - y_i$) as much as permitted by the constraint $y_i \geq 0$: thus $y_i = \max\{x_i - \mu, 0\}$, where $\mu$ is chosen so that $\sum_N y_i = r$.

In addition to these three methods, we can of course define a great variety of rationing methods, for instance fixed convex combinations of the three benchmarks. However such combinations violate consistency—an axiomatic requirement widely

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regarded as a compelling rationality property for fair division methods in a variety of contexts (including TU games, matching, assignment, etc.; see [Thomson 2005]).

A rationing method is consistent if, when we take away one agent from the set of participants, and subtract his share from the available resources, the division among the remaining set of claimants does not change. Formally,

\[ h_j(N \setminus \{i\}, x_{-i}, r - h_i(N, x, r)) = h_j(N, x, r) \text{ for all } j \neq i, \]

In the words of Balinski and Young [1982], “any part of a fair division should be fair.”

An important result [Young 1987] characterizes all rationing methods that are symmetric (w.r.t. individual claims), continuous (in \( x \) and \( r \)), and consistent. All such methods, and only those, have a \textit{parametric representation} as follows. Choose a continuous function \( \theta(x, \lambda) \) from \( \mathbb{R}_+^2 \) into \( \mathbb{R}_+ \), non decreasing in \( \lambda \) and such that \( \theta(x, 0) = 0 \) and \( \theta(x, \infty) = x \). Then divide \( r \) units of resources as follows:

\[ y_i = \theta(x_i, \lambda) \text{ for all } i, \text{ where } \lambda \text{ solves } \sum_N \theta(x_i, \lambda) = r. \] (1)

It is easy to see that the three benchmark methods are consistent and have such a representation.

Our goal in the paper [Moulin and Sethuraman 2011], and in subsequent forthcoming work, is to extend the rich theory of fair rationing to the more complex environment involving restrictions in the access to multiple types of resources.

2. BIPARTITE RATIONING AND CONSISTENCY

We consider the fair selection of a maximum flow in a bipartite network. The nodes on one side are agents, who wish to consume resources, which are the nodes on the other side. Not every agent has access to every resource, and such restrictions are captured by the bipartite graph. Each agent has a global demand, and the various resources are available in arbitrary quantities. We assume that the resources are substitutable, so that an agent cares only about the total amount of the resources he receives. A typical example is load balancing, where agents are workers, resources are types of jobs, and each agent is only qualified for certain jobs. See [Moulin and Sethuraman 2011] for more examples.

To motivate the consistency property in this context, we start with the example shown in Figure 1. Two agents Ann and Bob (nodes A and B on the left) have a

![Fig. 1. An example with 2 sources and 2 sinks](image-url)
Bipartite Rationing

In Figure 1a the bipartite graph is complete: every agent can access every resource node. Note that the resources are overdemanded: in any efficient allocation, all of the resources will be fully distributed. By symmetry Ann and Bob will receive 10 units each, which can be achieved by many different flows in the network.

Now assume Ann can access both a and b, whereas Bob can only access b (Figure 1b). It is still possible for Ann and Bob to receive 10 units each (by letting Ann receive only 2 units from b), and indeed this will be the egalitarian recommendation [Bochet et al. 2011]. Whether or not this is fair depends very much on our view of why Bob cannot access a. In this article, we postulate that agents ought to be held responsible for the compatibility constraints they impose. Therefore, as Ann and Bob have identical demands, but Ann is compatible with a larger set of resource nodes than Bob, she should receive more. More precisely Ann is entitled to all of the resource (8 units) at a, which she is the only one to claim. She still competes with Bob for the resources at b, but her claim on b is reduced from the original 12 to the residual claim of 12 − 8 = 4 units.

A version of consistency appropriate for bilateral rationing problems captures the idea that agents are responsible for their own compatibilities with the different types of resources. We can now take away either an agent or a type of resource: if the former, we subtract from each resource-type the share previously assigned to the departing agent; if the latter, we subtract from the claim of each agent the share of the departing resources he was previously receiving; in each case we insist that the division in the reduced problem remain as before. A stronger form of consistency can be applied to each edge of the graph: when we remove a certain edge, we subtract its flow from the capacity of both end nodes, and require as above that the solution choose the same flow in the reduced problem.

Our main goal is to understand which (and how) standard methods can be extended to bipartite methods while respecting consistency.

2.1 Model

A rationing problem involves a set $N$ of $n$ agents, a set $Q$ of $q$ types, and a bipartite graph $G \subseteq N \times Q$: an edge $(i,a) \in G$ indicates that agent $i$ can consume the type $a$. We define $f(i)$ to be the (non empty) set of types that $i$ is connected to, and $g(a)$ to be the (non empty) set of agents connected to type $a$. Each agent $i$ has a claim $x_i$ and each type $a$ has a capacity (amount it can supply) $r_a$. We assume that the resources are (weakly) overdemanded, so that we have a rationing situation. Formally, this means

$$\text{for all } B \subseteq Q: \ r_B \leq x_{g(B)}. \quad (2)$$

A bipartite rationing problem is a list $P = (N, Q, G, x, r)$ that satisfies (2). Let $\mathcal{P}$ denote the set of bipartite rationing problems.

A max-flow of $P$ is a flow $\phi = (\phi_{ia})_{(i,a) \in G} \in \mathbb{R}^G_+$ sending all the resources to the
agents, and such that no agent receives more than his demand:

\[ \phi_{g(a)} = r_a \text{ for all } a \in Q; \text{ and } \phi_{f(i)} \leq x_i \text{ for all } i \in N. \]

We write \( \mathcal{F}(P) \) for the set of max-flows in \( P \). A bipartite rationing method \( H \) associates to each problem \( P \in \mathcal{P} \) a max-flow \( \phi = H(P) \in \mathcal{F}(P). \)

We restrict attention to symmetric and continuous rationing methods: a method \( H \) is symmetric if the labels of the agents and resources do not matter; it is continuous if the mapping \( (x, r) \rightarrow H(G, x, r) \) is continuous in the relevant subset of \( \mathbb{R}^N_+ \times \mathbb{R}^Q_+ \). For brevity we use the informal definition of consistency given just before subsection 3.1: we speak of node-consistency when comparing a problem and its reduction after taking away an arbitrary agent or type, and of the stronger edge-consistency when we take away an arbitrary edge.

3. MAIN RESULTS

We find that the standard proportional method has a unique consistent extension to the bipartite case, whereas the two other benchmark methods have infinitely many such extensions.

We start with the proportional method. For any \( z \geq 0 \), define the function \( E_n(z) = z \ln(z) \), with the convention that \( E_n(0) = 0 \) (so the sum \( \sum \phi_{g(a)} \) is the familiar entropy of a vector \( z \)). Note that \( E_n(z) \) is strictly convex.

Given a problem \( P \in \mathcal{P} \), define \( \hat{\phi} \) as

\[ \hat{\phi} = \arg \min_{\phi \in \mathcal{F}(P)} \sum_{a \in G} E_n(\phi_{g(a)}) + \sum_{i \in N} E_n(x_i - \phi_{f(i)}) \] (3)

Problem (3) has a unique solution \( \hat{\phi} \) because the objective function is strictly convex and finite. We define the proportional method \( H^{pro} \) as \( H^{pro}(P) = \hat{\phi} \). This extends the standard proportional method: if \( G \) has exactly one resource node \( a \), the problem is to minimize \( \sum_{i \in N} E_n(y_i) + \sum_{i \in N} E_n(x_i - y_i) \) over all \( y \geq 0 \) such that \( y_N = r_a \), and the Kuhn-Tucker conditions say that \( \ln(\frac{y_i}{x_i - y_i}) \) is independent of \( i \).

Theorem 3.1. The proportional method \( H^{pro} \) is symmetric, continuous, and edge-consistent. Moreover, it is the only continuous and node-consistent method that is proportional for standard problems.

While the above characterization is useful, an equivalent definition sheds further light on the structure of the proportional method. This definition is restricted to the subset of irreducible problems, defined by the property that all the inequalities in (2) for \( B \not\subseteq Q \) are strict. These are the problems in which every edge in the graph carries a positive flow in some max-flow.

---

1We use the notation \( \phi_{g(a)a} = \sum_{i \in g(a)} \phi_{ia} \), and \( \phi_{f(i)} = \sum_{a \in f(i)} \phi_{ia} \).

2Any bipartite max-flow problem is the union of two separate rationing problems just described, one problem where all max-flows empty all resource nodes while agents are rationed, the other where max-flows fulfill all demands while resource nodes are not exhausted ([Moulin and Sethuraman 2011]). In this sense our theory applies to the selection of a fair max-flow in a general bipartite problem \( (N, Q, G, x, r) \) that may not satisfy (2).
Theorem 3.2. For any irreducible problem $P$ the system of equations in $z$:
\[
\sum_i z_i = x_N - r_Q \quad \text{and} \quad x_i = z_i + \sum_{a \in f(i)} \frac{z_i}{z_{g(a)}} r_a \quad \text{for all } i \in N
\]
has a unique solution $\hat{z} > 0$, and the proportional flow is
\[
\hat{\varphi}_{ia} = \frac{\hat{z}_i}{z_{g(a)}} r_a
\]
Note that $z_i$ is precisely agent $i$’s loss: $z_i = x_i - \hat{\varphi}_{i(f(i))}$. Hence a nice interpretation of Theorem 3.2: each resource is allocated proportionally to the agents who are connected to it, but the proportionality is with respect to the losses rather than the original claims. In the case of a standard method, losses $x_i - y_i$ are proportional to gains $y_i$, but this is not true any more in the bipartite context.

A straightforward generalization of Problem (3) delivers a large family of edge-consistent bipartite methods:

Theorem 3.3. Fix a strictly convex function $W$ and a convex function $B$, both from $\mathbb{R}_+^+$ into itself. For any problem $P \in \mathcal{P}$ the flow
\[
\hat{\varphi} = \arg \min_{\varphi \in \mathcal{F}(P)} \sum_{ia \in G} W(\varphi_{ia}) + \sum_{i \in N} B(x_i - \varphi_{i(f(i))})
\]
defines an edge-consistent, symmetric, and continuous bipartite rationing method.

The bipartite proportional method corresponds to $W = B = E_n$, and we know from Theorem 3.1, that this is the only edge-consistent extension of the standard proportional method. In contrast, the bipartite rationing methods in Theorem 3.3 contain infinitely many extensions of the uniform gains, and, in a limit sense, of the uniform losses methods.

By taking any strictly convex $W$ and $B \equiv 0$ in (5), we obtain a consistent extension of the uniform gains method. Formally:

Proposition 3.4. For any strictly convex function $W$ from $\mathbb{R}_+^+$ into itself, and any problem $P \in \mathcal{P}$, the flow
\[
\varphi = \arg \min_{\varphi \in \mathcal{F}(P)} \sum_{ia \in G} W(\varphi_{ia})
\]
defines a bipartite rationing method $H^W$ that is symmetric, continuous, edge-consistent and extends the uniform gains method for standard rationing problems. Different choices of $W$ yield infinitely many different methods $H^W$.

Turning finally to the uniform losses method, we proceed in two steps; first we take $W \equiv 0$ and for $B$ a strictly convex function in (5). Independently of the choice of $B$, this determines the net share $y_i = \varphi_{i(f(i))}$ of each agent $i$. Then we fix the edge-flows by one more minimization w.r.t. a strictly convex $W$ in (5). In the following statement we write $\mathcal{Y}(P)$ for the set of feasible net shares $y = (\varphi_{i(f(i))})_{i \in N}$ when $\varphi$ is a max-flow of $P$ ($\varphi \in \mathcal{F}(P)$).
Proposition 3.5. Fix any two strictly convex functions $W, B$ from $\mathbb{R}$ into itself. For any problem $P = (G, x, r) \in \mathcal{P}$, the net shares
\[ y = \arg \min_{y \in \mathcal{Y}(P)} \sum_{i \in N} B(x_i - y_i) \]
and the flow
\[ \varphi = \arg \min_{\varphi \in \mathcal{F}(P)} \sum_{ia \in G} W(\varphi_{ia}) \text{ where } P = (G, \varphi, r) \]
define a method $H^B \succ W$ that is symmetric, continuous, edge-consistent, and extends the uniform losses method for standard problems. The choice of $B$ does not matter, but different choices of $W$ yield infinitely many different methods.

4. TOWARD A GENERAL CHARACTERIZATION

It is puzzling that the proportional method admits a unique consistent extension, while uniform gains and uniform losses admit so many. Moreover in any consistent extension of uniform losses, the profile of net shares is fixed, whereas the various consistent extensions of uniform gains may yield different net shares.

A difficult, and more general, question is to understand which consistent standard methods can be extended at all to the bipartite framework as continuous, symmetric and node- (or edge-) consistent methods.

Some partial understanding of these questions follows from generalizing the system (4) in Theorem 3.2, where the unknown is the profile of losses $z_i$. To explain this approach, we first go back to the standard rationing model and note that a large subset of parametric methods (section 1) can be described in loss format as follows. System (1) explains how we compute shares given the profile of demands.

Can we similarly compute shares if we know the profile of losses $z_i$? This amounts to asking if the following system in $y, \lambda$
\[ y_i = \theta(z_i + y_i, \lambda) \text{ for all } i, \text{ and } \sum_N y_i = r, \]
has a unique solution for a given vector $z$.

We impose two mild properties that all standard rationing methods discussed in the literature satisfy:

- **Ranking (RK):** $x_i \leq x_j \Rightarrow y_i \leq y_j$
- **Ranking* (RK*):** $x_i \leq x_j \Rightarrow x_i - y_i \leq x_j - y_j$

In the parametric format this amounts to saying that $\theta(x, \lambda)$ is weakly increasing and 1-Lipschitz in $x$. If in addition the method satisfies the strict version of RK*:

- **Strict Ranking* (SRK*):** $x_i < x_j \Rightarrow x_i - y_i < x_j - y_j$, then for any $z_i, \lambda$ the equation $y_i = \theta(z_i + y_i, \lambda)$ has at most one solution $y_i = \beta(z_i, \lambda)$, and the function $\beta$ is continuous and weakly increasing in both variables.

It is the loss-parametrization of our method:

- given $z$, the shares are $y_i = \beta(z_i, \lambda)$ where $\lambda$ solves $\sum_N \beta(z_i, \lambda) = r$. 

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The proportional method admits the gain-parametrization \( y_i = \theta(x_i, \lambda) = \frac{\lambda x_i}{x_i + \lambda} \), equivalent to its loss-parametrization \( \beta(z_i, \lambda) = \lambda z_i \).

The uniform gains and uniform losses method do not meet SRK*, and the equation \( y_i = \theta(z_i + y_i, \lambda) \) may indeed have a full interval of solutions \( y_i \), so these two methods are excluded from the analysis below.

But the rich families of parametric methods known as equal sacrifice, and dual equal sacrifice [Young 1988], all admit loss-parametrizations. For instance the following equal sacrifice methods

\[ h^1: 1 \cdot y_i - 1 \cdot x_i = 1 \Rightarrow \theta(x_i, \lambda) = \frac{x_i}{x_i + \lambda} \]

has the loss-parametrization

\[ \beta(z_i, \lambda) = \frac{\lambda z_i}{(\frac{z_i^2}{4} + \lambda z_i)^{\frac{1}{2}} + \frac{z_i}{2}} \]

and the dual equal sacrifice method \( h^2 \)

\[ h^2: 1 \cdot x_i - y_i - 1 \cdot x_i = \lambda \Rightarrow \theta(x_i, \lambda) = \frac{x_i^2}{1 + \lambda x_i} \]

has the loss-parametrization

\[ \beta(z_i, \lambda) = \frac{\lambda z_i^2}{1 - \lambda z_i} \]

The latter example shows that the function \( \beta \) may not be defined on the entire orthant \( \mathbb{R}^2_+ \), a feature that brings only minor technical complications.

Fix now a standard method \( h \) in the loss format \( \beta(z_i, \lambda) \), and suppose that it can be extended to a consistent bipartite method \( H \). Consider a strictly overdemanded problem \( P = (N, Q, G, x, r) \), i.e., such that the inequalities in (2) are strict for all \( B \), including \( B = Q \). Then the flow \( H(P) = \varphi \) is

\[ \varphi_{ia} = \beta(z_i, \lambda^a) \text{, for all } ia \in G, \]

where the profiles of losses \( z \in \mathbb{R}^N_+ \) and of parameters \( \lambda \in \mathbb{R}^Q_+ \) uniquely solve the following system

\[ x_i = z_i + \sum_{a \in f(i)} \beta(z_i, \lambda^a) \text{ for all } i \in N \quad (7) \]

\[ \sum_{i \in g(a)} \beta(z_i, \lambda^a) = r_a \text{ for all } a \in Q \quad (8) \]

Indeed agent \( i \)'s loss \( z_i \) is defined by \( x_i = z_i + \sum_{a \in f(i)} \varphi_{ia} \). In repeated applications of consistency, eliminating all resources but \( a \), the loss profile is preserved, therefore when we are left with a single resource, its flow is derived from the loss-parametrization. Note that system (7) and (8) boils down to (4) for the proportional method.

This result shows that under the assumption SRK*, a standard rationing method admits no more than one consistent bipartite extension, and this extension is well defined for all strictly overdemanded problems. The difficult open question is...
whether or not we can extend continuously the method $H$ to all irreducible, then to all overdemanded problems. If the answer to the latter is positive, we will conclude that all standard methods meeting SRK* have a unique consistent extension to the bipartite setting.

5. DIRECTIONS FOR FUTURE RESEARCH

All reasonable symmetric standard rationing methods, $y_i = h(x_i, r)$, including the three benchmark methods, meet the following monotonicity properties (in addition to RK and RK*):

1. Resource Monotonicity (RM): $r \rightarrow y_i$ is weakly increasing for all $i$
2. Claim Monotonicity (CM): $x_i \rightarrow y_i$ is weakly increasing for all $i$
3. Cross Monotonicity (CRM): $x_i \rightarrow y_j$ is weakly decreasing for all $j \neq i$

The corresponding properties for bipartite rationing methods are equally compelling. Properties CM and CRM have the same definition, and can be strengthened as follows:

- Edge-CM: $x_i \rightarrow \varphi_{ia}$ is weakly increasing for all $i$ and all $a \in f(i)$
- Edge-CRM: $x_i \rightarrow \varphi_{ja}$ is weakly decreasing for all $j \neq i$ and $a \in f(i) \cap f(j)$

Then RM, RK, and RK* are adapted as follows:

- RM: $r_a \rightarrow y_i$ is weakly increasing for all $i \in g(a)$
- RK: $x_i \leq x_j \Rightarrow y_i \leq y_j$ for all $i, j$ such that $f(i) = f(j)$
- RK*: $x_i \leq x_j \Rightarrow x_i - y_i \leq x_j - y_j$ for all $i, j$ such that $f(i) = f(j)$

We have shown that system (7) and (8) implies the properties CM, CRM, RK, and RK*, but cannot say anything about any of these other properties, even for the bipartite proportional method. For the extensions of uniform gains and uniform losses, it is possible that the answer depends upon the choice of the convex function $W$.

Another interesting direction for future research is to study similar questions in a strategic framework. Consider an environment in which agents have single-peaked preferences over their allocations. The peaks (or more generally, preferences) are not known to the mechanism designer, who must design a strategyproof mechanism to allocate the resources to the agents. This is a well-studied problem in the standard rationing model [Sprumont 1991]; in particular, the mechanism assigning to each agent their allocation under the uniform-gains method is strategyproof, in the sense that no agent has an incentive to misreport their peaks (and the mechanism only works with the information about the peaks of the agents). It is a happy coincidence that this method is also consistent. In a recent series of papers, Bochet et al. [2011; 2012] study these questions in the bipartite framework and propose the egalitarian mechanism that generalizes the uniform gains method to the bipartite setting. Unfortunately the method underlying their mechanism fails consistency. A natural open question is to design a strategyproof mechanism in this setting such that the associated method is node or edge-consistent.

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3They also admit an “edge” version, omitted for brevity.
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The Price of Anarchy in Games of Incomplete Information

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We outline a recently developed theory for bounding the inefficiency of equilibria in games of incomplete information, with applications to auctions and routing games.

Categories and Subject Descriptors: F.0 [Theory of Computation]: General
General Terms: Algorithms, Economics, Theory
Additional Key Words and Phrases: Price of anarchy, incomplete information, congestion games, auctions

1. INTRODUCTION

Every student of game theory learns early and often that equilibria are inefficient. Such inefficiency is ubiquitous, and is present in many real-world situations and for many different reasons: in Prisoner’s Dilemma-type scenarios; from uninternalized negative externalities in the tragedy of the commons and in games with congestion effects; from uninternalized positive externalities with a public good or with network effects; from a failure to coordinate in team games; and so on.

The past ten years have provided an encouraging counterpoint to this widespread equilibrium inefficiency: in a number of interesting application domains, game-theoretic equilibria provably approximate the optimal outcome. Phrased in modern jargon, the price of anarchy — the worst-case ratio between the objective function value of an equilibrium, and of optimal outcome — is close to 1 in many interesting games.

Essentially all initial work on the price of anarchy studied full-information games, where all players’ payoffs are assumed to be common knowledge. Now that the study of equilibrium inefficiency has grown in scope and considers strategically interesting auctions and mechanisms, there is presently a well-motivated focus on the price of anarchy in games of incomplete information, where players are uncertain about each others’ payoffs. A recent paper [Roughgarden 2012], outlined below, develops tools for bounding the price of anarchy in such games.

2. EXECUTIVE SUMMARY: PRICE OF ANARCHY BOUNDS FOR BAYES-NASH EQUILIBRIA VIA EXTENSION THEOREMS (I.E., WITHOUT THE PAIN)

Pure-strategy Nash equilibria — where each player deterministically picks a single action — are often easy to reason about. For this reason, the price of anarchy of
a game is often analyzed, at least initially, only for the game’s pure-strategy Nash equilibria. But as much as he or she might want to, the conscientious researcher cannot stop there. Performance guarantees for more general classes of equilibria are crucial for several reasons (for example, pure-strategy Nash equilibria need not exist). In particular, a fundamental assumption behind the Nash equilibrium concept is that all players’ preferences are common knowledge, and this assumption is violated in most auction and mechanism design contexts, where participants have private information.

Extending price of anarchy bounds beyond pure Nash equilibria is an extremely well motivated activity, but it is also potentially dispiriting, for two reasons. The first is that the analysis generally becomes more complex, with one or more unruly probability distributions obfuscating the core argument. The second is that enlarging the set of permissible equilibria can only degrade the price of anarchy (which is a worst-case measure). Thus the work can be difficult, and the news can only be bad.

Can we obtain price of anarchy bounds for more general equilibrium concepts without working any harder than we already do to analyze pure-strategy Nash equilibria? Ideal would be an extension theorem that could be used in the following “black-box” way: (1) prove a bound on the price of anarchy of pure-strategy Nash equilibria of a game; (2) invoke the extension theorem to conclude immediately that the exact same approximation bound applies to some more general equilibrium concept. Such an extension theorem would dodge both potential problems with generalizing price of anarchy bounds beyond pure Nash equilibria — no extra work, and no loss in the approximation guarantee.

Since there are plenty of games in which (say) the worst mixed-strategy Nash equilibrium is worse than the worst pure-strategy Nash equilibrium (like “Chicken”), there is no universally applicable extension theorem of the above type. The next-best thing would be an extension theorem that applies under some conditions — perhaps on the game, or perhaps on the method of proof used to bound the price of anarchy of pure Nash equilibria. If such an extension theorem existed, it would reduce proving price of anarchy bounds for general equilibrium concepts to proving such bounds in a prescribed way for pure-strategy Nash equilibria.

The first example of such an extension theorem was given in [Roughgarden 2009], for full-information games. The key concept in [Roughgarden 2009] is that of a smooth game. Conceptually, a full-information game is smooth if the objective function value of every pure-strategy Nash equilibrium $a$ can be bounded using the following minimal recipe:

1. Let $a^*$ denote the optimal outcome of the game.
2. Invoke the Nash equilibrium hypothesis once per player, to derive that each player $i$’s payoff in the Nash equilibrium $a$ is at least as high as if it played $a_i^*$ instead. Do not use the Nash equilibrium hypothesis again in the rest of the proof.
3. Use the inequalities of the previous step, possibly in conjunction with other properties of the game’s payoffs, to prove that the objective function value of $a$ is at least some fraction of that of $a^*$.

Many interesting price of anarchy bounds follow from “smoothness proofs” of this
The main extension theorem in Roughgarden [2009] is that every price of anarchy bound proved in this way — seemingly only for pure Nash equilibria — automatically extends to every mixed-strategy Nash equilibrium, correlated equilibrium, and coarse correlated equilibrium of the game.

Our new paper [Roughgarden 2012] presents a general extension theorem for games of incomplete information, where players’ private preferences are drawn independently from prior distributions that are common knowledge. This extension theorem reduces, in a “black-box” fashion, the task of proving price of anarchy bounds for mixed-strategy Bayes-Nash equilibria to that of proving such bounds in a prescribed way for pure-strategy Nash equilibria in every induced game of full information (after conditioning on all players’ preferences). With this extension theorem, one can prove equilibrium guarantees for games of incomplete information without ever leaving the safe confines of full-information games.

We conclude this section with an overview of the main points of the paper [Roughgarden 2012].

1 Smooth games of incomplete information are defined. The definition is slightly stronger, in a subtle but important way, than requiring that every induced full-information game is smooth.

2 There is an extension theorem for smooth games of incomplete information: price of anarchy bounds for pure Nash equilibria for all induced full-information games extend automatically to all mixed-strategy Bayes-Nash equilibria with respect to a product prior distribution over players’ preferences.

3 Many games of incomplete information for which the price of anarchy has been studied are smooth in this sense. Thus this extension theorem unifies much of the known work on the price of anarchy in games of incomplete information.

4 This extension theorem implies new bounds on the price of anarchy of Bayes-Nash equilibria in congestion games with incomplete information.

5 For Bayes-Nash equilibria in games with correlated player preferences, there is no general extension theorem for smooth games. (Additional conditions under which the extension can be recovered are given in Caragiannis et al. [2012].)

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Some of our results were also obtained, subsequently but independently, in [Syrgkanis 2012]. There are also results in [Syrgkanis 2012] for mechanisms with first-price payment rules, which are not considered in [Roughgarden 2012].
We present a model for routing games in which edge delay functions are uncertain and users are risk-averse. We investigate how the uncertainty and risk-aversion transform the classical theory on routing games, including equilibria existence, characterization and price of anarchy.

Categories and Subject Descriptors: J.4 [Computer Applications]: Social and Behavioral Sciences—Economics

General Terms: Algorithms, Economics, Theory

Additional Key Words and Phrases: Routing games, Wardrop Equilibria, Uncertainty, Risk-aversion, Mean-risk objective

1. INTRODUCTION

Routing games were one of the central examples in the development of algorithmic game theory. In these games, multiple users need to route between different source-destination pairs and edges are congestible, namely, each edge delay \( l_e(x) \) is a non-decreasing function of the flow or number of users \( x \) on the edge. Many of the fundamental game theoretic questions are now well understood for these games, for example, does equilibrium exist, is it unique, can it be computed efficiently, does it have a compact representation; the same questions can be asked of the socially optimal solution that minimizes the total user delay. Furthermore, routing games were a primary motivation and application for the study of the price of anarchy, which quantifies the inefficiency of equilibria.

So far, most research has focused on the classical models in which the edge delays are deterministic. In contrast, real world applications contain a lot of uncertainty, which may stem from exogenous factors such as weather, time of day, weekday versus weekend, etc. or endogenous factors such as the network traffic. Furthermore, many users are risk-averse in the presence of uncertainty, so that they do not simply want to minimize expected delays and instead may need to add a buffer to ensure a guaranteed arrival time to a destination. This fundamentally changes the mathematical structure of the routing problems and, consequently, the behavior and properties in routing games as well.

Our work [Nikolova and Stier-Moses 2011; 2012] aims to initiate a theoretical study of how uncertainty and risk aversion transform the classical theory of routing games. Integrating different models of uncertainty and different measures of risk can easily fill a long-term research agenda. We therefore focus on the special model...
defined in the next section and hope to motivate other researchers to join in the effort by considering generalizations or alternative models.

2. MODEL

Consider a directed graph $G = (V, E)$ with an aggregate demand of $d_k$ units of flow between source-destination pairs $(s_k, t_k)$ for $k \in K$. We let $P_k$ be the set of all paths between $s_k$ and $t_k$, and $P := \cup_{k \in K} P_k$ be the set of all paths. We encode players decisions as a flow vector $f = (f_\pi)_{\pi \in P} \in \mathbb{R}^{|P|}$ over all paths. Such a flow is feasible when demands are satisfied, as given by constraints $\sum_{\pi \in P_k} f_\pi = d_k$ for all $k \in K$. The congestible network is modeled with stochastic delay functions $\ell_e(x_e) + \xi_e(x_e)$ for each edge $e \in E$. Here, $\ell_e(x_e)$ measures the expected delay when the edge has flow $x_e$, and the random variable $\xi_e(x_e)$ represents the stochastic delay error.

The function $\ell_e(\cdot)$ is assumed continuous and non-decreasing, $\mathbb{E}(\xi_e(x_e)) = 0$, and $\text{Stdev}(\xi_e(x_e)) = \sigma_e(x_e)$, for a continuous function $\sigma_e(\cdot)$. Although the distribution of delay may depend on the flow $x_e$, we separately consider the simplified case in which $\sigma_e(x_e) = \sigma_e$ is a constant given exogenously, independent from $x_e$. We also assume that delays are uncorrelated with each other (see [Nikolova 2009], p. 96 for a discussion on how to incorporate local correlations).

Risk-averse players choose paths according to a mean-stdev objective, which we refer to as the cost along route $\pi$:

$$Q_\pi(f) := \sum_{e \in \pi} \ell_e\left(\sum_{p \in p} f_p\right) + \gamma \sqrt{\sum_{e \in \pi} \sigma_e\left(\sum_{p \in p} f_p\right)^2},$$  \hspace{1cm} (1)$$

where $\gamma \geq 0$ quantifies the risk aversion of players, assumed homogeneous.

Adding a constant number of standard deviations to the expectation of delay is a natural approach for adding a buffer to increase the reliability of a route. A compelling interpretation of this objective in the case of normally-distributed uncertainty is that the mean-stdev of a path equals a percentile of delay along it. This model is also related to typical quantifications of risk, most notably the value-at-risk objective commonly used in finance, whereby one seeks to minimize commute time subject to arriving on time to a destination with at least, say, 95% chance. The mean-stdev risk measure has also been used by transportation practitioners, who base the definition of the travel time reliability index on it [Schränk et al. 2010]. At the same time, the measure has been criticized for sometimes preferring a strictly dominated solution.\(^1\) In essence, the objective has a preference for more certain routes, which in fact may be an advantage in applications such as telecommunication networks (voice or video streaming), transportation (intercity bus routes), task planning, robot motion planning, etc. Further discussion of the objective can be found in our work [Nikolova and Stier-Moses 2011; 2012].

\(^1\)For instance, between choosing a path that always takes 1 hr vs a path that takes 1 hr or 50 min with probability $\frac{1}{2}$ each, the objective may prefer the first path even though it is stochastically dominated, since the second path is penalized for its variability through the standard deviation term in the objective.
3. RESULTS

We generalize the traditional model of Wardrop competition [Wardrop 1952] by incorporating stochastic travel times. Technically, this model is much harder to analyze than the traditional one because it is non-additive, namely the cost of a path is not equal to the sum of costs of edges along the path [Gabriel and Bernstein 1997]. This in turn means that an equilibrium in the stochastic setting does not decompose to equilibria in subnetworks of the given network, leading to computational and structural complications. Depending on the specific details of the application one has in mind, users may be small or large [Harker 1988]. We consider both infinitesimal users, referred to as the non-atomic case, as well as users that control a strictly positive demand, referred to as the atomic case.

To analyze the problem and to establish the existence of equilibrium, we draw from a diverse spectrum of tools from potential games and convex analysis to the theory of variational inequalities and nonconvex (stochastic) shortest paths. We consider four settings of nonatomic vs. atomic users and exogenous vs. endogenous variability of travel times. Our conclusions and methods are different in each of these settings. In the nonatomic case with standard deviation of travel times given exogenously, we prove that equilibria always exist using a convex problem with exponentially-many variables similar to that of [Ordóñez and Stier-Moses 2010].

The atomic case with exogenous standard deviations is shown to be a potential game and therefore a pure-strategy Nash equilibrium always exists. To characterize the equilibria of the nonatomic version of the problem when the standard deviations of travel times are endogenous, we use a variational inequality formulation [Hartman and Stampacchia 1966; Smith 1979; Dafermos 1980] that draws ideas from the nonlinear complementary problem formulation of [Aashtiani and Magnanti 1981]. In this case, an equilibrium always exists; in fact, not only for our specific mean-stdev objective but also for any general continuous objective. In contrast, the atomic case with endogenous standard deviation does not always admit a pure-strategy Nash equilibrium. We summarize these results in Table I.

Next, we investigate if there is a succinct representation (in terms of a small set of paths) of user and system-optimal flows in the case of non-atomic users with stochastic travel times. Our results here are independent of whether the standard deviations are exogenous or endogenous. We prove that if one is given a solution (either a Wardrop equilibrium or a system optimum) as an edge-flow, not every path decomposition is a solution, in contrast to the deterministic case where every decomposition works. Nevertheless, there is always a succinct solution that uses at most $|E| + |K|$ paths, where $E$ is the set of edges in the network and $K$ is the set of origin-destination pairs. Although the complexity of computing a solution is

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<td>Nonatomic Users</td>
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<td>(exponentially-large convex program)</td>
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<td>Atomic Users</td>
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Table I. Existence of equilibria in mean-risk stochastic selfish routing games.
unknown (actually, even the complexity of computing a single stochastic shortest path is unknown), this result says that there is some hope because at least solutions can be efficiently encoded.

Finally, we analyze the price of anarchy of mean-risk Wardrop equilibria under stochastic travel times with respect to the socially-optimal solution, for the case of nonatomic users. The social optimum is defined as the flow minimizing the total cost incurred by users, as given by their mean-stdev objective. Surprisingly, under exogenous standard deviations, uncertainty and risk aversion do not exacerbate the inefficiency of equilibria. The price of anarchy remains equal to that of deterministic nonatomic games. Namely, it is $4/3$ for the case of linear expected travel times [Roughgarden and Tardos 2002] and $(1 - \beta(L))^{-1}$ for an appropriately defined constant $\beta(L)$ for expected travel time functions in a class $\mathcal{L}$ [Roughgarden 2003; Correa et al. 2004; 2008].

The case of endogenous standard deviations presents a significant additional difficulty that makes the square root terms in different paths interrelated functions of the path flow that cannot be analyzed separately; a general price of anarchy bound for this case remains elusive. Nevertheless, we show that, despite the square root term, the path costs are convex whenever the individual travel times and standard deviations on edges are convex. Consequently, we present sufficient conditions for convexity of the social cost, which are similar to the sufficient conditions for uniqueness of equilibrium in its variational inequality characterization. Unfortunately, these conditions are fragile and in general the social cost will not be convex and may admit a non-connected set of multiple global minima but we can still identify settings where the price of anarchy is 1.

4. OPEN PROBLEMS

From a high-level philosophical perspective, it is intriguing to understand how users make decisions and what are the right risk-aversion models in uncertain settings. For the correct modeling of routing and other games studied in Algorithmic Game Theory and Mechanism Design, it would be beneficial to draw from fields and areas that have a tradition in decision-making under uncertainty such as Expected Utility Theory and alternative Non-Expected Utility Theories (considered at the intersection of psychology and economics), Portfolio Optimization, Operations Research and Finance.

Some concrete questions that arise from our work include:

—What is the complexity of computing an equilibrium when it exists (exogenous standard deviations with atomic or nonatomic players; endogenous standard deviations with nonatomic players)?

—What is the complexity of computing the socially-optimal solution? What is the complexity of computing the socially-optimal flow decomposition if one knows the edge-flow that represents a socially-optimal solution?

—Can there be multiple equilibria in the nonatomic game with endogenous standard deviations?

—What is the price of anarchy for stochastic Wardrop equilibria in the setting of nonatomic games with endogenous standard deviations, for general graphs and general classes of cost functions?
—Can some of the results in this paper be extended to the case of users with heterogenous attitudes toward risk [Ordóñez and Stier-Moses 2010]?

Of course, one could pursue other natural models and player objectives and build upon or complement the theory we have developed here. In particular, our model might be enriched by also considering stochastic demands to make the demand side more realistic.

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Computing Approximate Pure Nash Equilibria in Congestion Games

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Among other solution concepts, the notion of the pure Nash equilibrium plays a central role in Game Theory. Pure Nash equilibria in a game characterize situations with non-cooperative deterministic players in which no player has any incentive to unilaterally deviate from the current situation in order to achieve a higher payoff. Unfortunately, it is well known that there are games that do not have pure Nash equilibria. Furthermore, even in games where the existence of equilibria is guaranteed, their computation can be a computationally hard task. Such negative results significantly question the importance of pure Nash equilibria as solution concepts that characterize the behavior of rational players. Approximate pure Nash equilibria, which characterize situations where no player can significantly improve her payoff by unilaterally deviating from her current strategy, could serve as alternative solution concepts provided that they exist and can be computed efficiently. In this letter, we discuss recent positive algorithmic results for approximate pure Nash equilibria in congestion games.

Categories and Subject Descriptors: F.2 [Theory of Computation]: Analysis of Algorithms and Problem Complexity; J.4 [Computer Applications]: Social and Behavioral Sciences-Economics
General Terms: Algorithms, Economics, Theory
Additional Key Words and Phrases: Pure Nash equilibria, potential games

1. PROBLEM STATEMENT AND RELATED WORK

In a weighted congestion game, players compete over a set of resources. Each player has a positive weight. Each resource incurs a latency to all players that use it; this latency depends on the total weight of the players that use the resource according to a resource-specific, non-negative, and non-decreasing latency function. Among a given set of strategies (over sets of resources), each player aims to select one selfishly, trying to minimize her individual total cost, i.e., the sum of the latencies on the resources in her strategy. Typical examples include congestion games in

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networks, where the network links correspond to the resources and each player has alternative paths that connect two nodes as strategies.

The case of unweighted congestion games (i.e., when all players have unit weight) has been widely studied in the literature. Rosenthal [1973] proved that these games admit a potential function with the following remarkable property: the difference in the potential value between two states (i.e., two snapshots of strategies) that differ in the strategy of a single player is equal to the difference of the cost experienced by this player in these two states. This immediately implies the existence of a pure Nash equilibrium. Any sequence of improvement moves by the players strictly decreases the value of the potential and a state corresponding to a local minimum of the potential will eventually be reached; this corresponds to a pure Nash equilibrium. For weighted congestion games, potential functions are known only in special cases (e.g., when the latency functions are linear; see [Fotakis et al. 2005]). Actually, in games with polynomial latency functions (e.g., quadratic), pure Nash equilibria may not even exist [Goemans et al. 2005].

Potential functions provide only inefficient proofs of existence of pure Nash equilibria. Fabrikant et al. [2004] proved that the problem of computing a pure Nash equilibrium in a (unweighted) congestion game is \text{PLS}-complete (informally, as hard as it could be, given that there is an associated potential function). One consequence of \text{PLS}-completeness results is that almost all states in some congestion games are such that any sequence of players’ improvement moves that originates from these states and reaches pure Nash equilibria is exponentially long. Efficient algorithms are known only for special cases. For example, Fabrikant et al. [2004] show that the Rosenthal’s potential function can be (globally) minimized efficiently by a flow computation in unweighted congestion games in networks when the strategy sets of the players are symmetric.

The above negative results have led to the study of the complexity of approximate pure Nash equilibria. A $\rho$-approximate (pure Nash) equilibrium is a state, from which no player has an incentive to deviate so that she decreases her cost by a factor larger than $\rho$. The only positive result that appeared before our recent work is due to Chien and Sinclair [2011] and applies to symmetric unweighted congestion games: under mild assumptions on the latency functions and on the participation of the players in the dynamics, the $(1+\epsilon)$-improvement dynamics converges to a $(1+\epsilon)$-approximate equilibrium after a polynomial number of steps. For non-symmetric (unweighted) congestion games with more general latency functions, Skopalik and Vöcking [2008] show that the problem is still \text{PLS}-complete for any polynomially computable $\rho$.

2. OUR CONTRIBUTION

In two recent papers [Caragiannis et al. 2011; 2012], we present algorithms for computing $O(1)$-approximate equilibria in unweighted and weighted non-symmetric congestion games with polynomial latency functions of constant maximum degree. Our algorithm for unweighted congestion games is presented in [Caragiannis et al. 2011]. It computes $(2+\epsilon)$-approximate pure Nash equilibria in games with linear latency functions, and $d^{d+o(d)}$ approximate equilibria for polynomial latency functions of maximum degree $d$. The algorithm is surprisingly simple. Essentially,
starting from an initial state, it computes a sequence of best-response player moves of length that is bounded by a polynomial in the number of players and $1/\epsilon$. The sequence consists of phases so that the players that participate in each phase experience costs that are polynomially related. This is crucial in order to obtain convergence in polynomial time. Another interesting part of our algorithm is that, within each phase, it coordinates the best response moves according to two different (but simple) criteria; this is the main tool that guarantees that the effect of a phase to previous ones is negligible and, eventually, an approximate equilibrium is reached.

In [Caragiannis et al. 2012], we significantly extend our techniques and obtain an algorithm that computes $O(1)$-approximate equilibria in weighted congestion games. For games with linear latency functions, the approximation guarantee is $3 + \sqrt{5} + \epsilon$ for arbitrarily small $\epsilon > 0$; for latency functions of maximum degree $d \geq 2$, it is $d^d + o(d)$. These results are much more surprising than they look at first glance. Given that weighted congestion games with superlinear latency functions do not admit potential functions, it is not even clear that $O(1)$-approximate equilibria exist. In order to bypass this obstacle, we introduce a new class of potential games (that we call $\Psi$-games), which "approximate" weighted congestion games with polynomial latency functions in the following sense. $\Psi$-games of degree 1 are linear weighted congestion games. Each weighted congestion game of degree $d \geq 2$ has a corresponding $\Psi$-game of degree $d$ defined in such a way that any $\rho$-approximate equilibrium in the latter is a $d!\rho$-approximate equilibrium for the former. As an intermediate new result, we obtain that weighted congestion games with polynomial latency functions of degree $d$ have $d!$-approximate equilibria. Our algorithm is actually applied to $\Psi$-games; it has a simple general structure similar to our algorithm for unweighted games but has also important differences that are due to the dependency of the cost of each player on the weights of other players. Again, the algorithm essentially identifies a best-response sequence of player moves in the $\Psi$-game that leads to an approximate equilibrium; its length is now polynomial in terms of the number of bits in the representation of the game and $1/\epsilon$.

In both cases, the approximation guarantee is marginally higher than a quantity that characterizes the potential function of the game; this quantity (which we call the stretch) is defined as the worst-case ratio of the potential value at an almost exact pure Nash equilibrium over the globally optimum potential value. For example, the stretch is almost 2 for linear unweighted congestion games, $3 + \sqrt{5}$ for linear weighted congestion games, and $d^d + o(d)$ for $\Psi$-games of degree $d \geq 2$. A more thorough literature review, the detailed description of the algorithms, and the analysis details can be found in [Caragiannis et al. 2011; 2012].

REFERENCES


How to Approximate Optimal Auctions

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Bayesian auction design investigates how to sell scarce resources to agents with private values drawn according to known distributions. A natural objective in this setting is revenue maximization. The seminal work of Roger Myerson presents a recipe for designing revenue-maximizing auctions in single-parameter settings. Myerson defines a function that maps each agent’s value to a new number called the virtual value. The optimal auction then chooses the solution with maximum virtual surplus subject to the feasibility constraints. Often this underlying optimization problem is NP-hard and is complicated by the fact that virtual values may be negative. Prior work approximates the optimal auction by approximating this optimization problem. Such a solution can be interpreted as approximating the optimal auction point-wise, i.e., for each realization of values the revenue of the derived auction is close to that of the Myerson auction. In this letter, we suggest a different approach:

Approximate the maximum virtual surplus on average with respect to the induced virtual value distributions.

Our approach has the advantage that sometimes the point-wise optimization problem is not approximable to within any reasonable factor. However, by leveraging the fact that any distribution of virtual values has non-negative expectation, it is sometimes possible to get good average-case approximations. Furthermore, the optimal auction is itself an average-case guarantee: it maximizes the revenue on average with respect to the distributions, but it may lose substantial revenue on certain value profiles. Thus our average-case guarantee is without loss of generality.

We showcase our approach using the problem of selling goods with positive network externalities. For this problem, the underlying optimization problem is inapproximable, yet with our approach we are able to prove that a natural greedy auction is a constant-factor approximation.

Categories and Subject Descriptors: J.4 [Social and Behavior Sciences]: Economics
General Terms: Algorithms, Economics, Theory

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1. INTRODUCTION

Auctions are often used to allocate resources to agents. In the most simple auction, a single item, say a painting, is sold to a single agent. The seller is unsure of the exact value of the agent, but knows the distribution of the agent’s value. Based on this distribution, the seller sets a price that maximizes his expected revenue with respect to the value distribution. For example, if the agent’s value is uniformly distributed in the interval \([0,1]\), then for a fixed price \(p \in [0,1]\), the expected revenue is \(p(1-p)\). A price of \(p = 1/2\) achieves the maximum expected revenue of 1/4 for this setting. For any distribution, the solution to this simple setting is a straight-forward maximization program.

For more complex settings with multiple agents and potentially complicated constraints on the set of feasible outcomes, the seminal paper by Myerson [Myerson 1981] fully characterizes auctions that maximize revenue in expectation over the value distributions. By this characterization, the expected revenue of any auction is equal to the total expected virtual value \(\phi(v) = v - \frac{1-F(v)}{f(v)}\) of the allocated agents, where \(F(.)\) is the cumulative distribution function and \(f(.)\) is the corresponding density function of the value distribution.\(^1\) Thus the auction that maximizes expected virtual value is optimal. For the example above, the virtual value function is \(\phi(v) = 2v - 1\), and so the auction that allocates to the agent whenever his virtual value is positive has revenue of \((1/2)E[2v - 1|v > 1/2] = 1/4\), matching the above pricing scheme as expected.

In many applications, computing the maximum expected virtual value is computationally hard due to the feasibility constraints on the set of winners. In such settings, the best known computationally tractable auctions first design an \(\alpha\)-approximation algorithm for the underlying virtual value optimization problem, and then use this solution to achieve an \(\alpha\) fraction of the optimal revenue. Unfortunately, for many settings, there is provably no good worst-case approximation algorithm for the relevant optimization problem. In particular, for feasibility constraints that are not downward-closed\(^2\), the fact that virtual values may be negative often implies hardness of approximation results. However, an approximately optimal auction need not provide worst-case guarantees: an average-case guarantee is sufficient. This is because the optimal auction anyway provides an optimal average-case guarantee, and there is no mechanism with high revenue for every instantiation of values.

This observation seems useless at first glance: a worst-case instance can be simulated by an average-case one where the distribution is a point mass. However, virtual value distributions are not arbitrary. They are guaranteed to have non-negative expectation, suggesting that non-downward-closed problems that are hard in the worst-case may be approximable on average. This fact enables us to design algorithms that perform well on average with respect to any distribution whose expectation is non-negative. Such algorithms imply approximately optimal auctions.

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\(^1\)Note the prices are not explicit in this characterization; they are implied by the incentive constraints. To gain intuition for this, observe, as the instantiation of values are known only to agents, prices can not always extract full surplus or otherwise agents would misreport their values.

\(^2\)Feasibility constraints are downward-closed if any subset of feasible winners is also feasible.
2. SHOWCASE PROBLEM

In prior work [Haghpanah et al. 2011], we use our technique to design auctions in social networks for goods that exhibit single-parameter submodular network externalities in which a bidder’s value for an outcome is a fixed private type times a known submodular function of the allocation of his friends. Our main result considers step-function externalities in which a bidder’s value for an outcome is either zero, or equal to his private type if at least one friend has the good. In this setting, the virtual value optimization problem equates to finding, in a vertex-weighted graph with possibly negative vertex weights, a maximum-weight subset of vertices whose induced subgraph has no singleton components. We observe via reduction to set-buying that approximating this optimization problem within even a linear factor on every sampling of the values is NP-hard. Even on average, we prove that our problem remains APX-hard. However, we are able to design constant approximations for several versions of the problem.

We first note that there is a simple \((1/2)\)-approximation for our problem. The algorithm divides the graph into two subsets of vertices, such that each vertex in each set has a neighbor in the other. This can be done, for example, by constructing a spanning tree of the graph and then taking a bipartite partitioning of it. The allocation strategy is to then pick the set with better expected revenue (computed with samples from the value distributions and without looking at the instantiation of values), extract revenue from that set (by allocating to each agent in the set with positive virtual value), and allocate to agents in the other set in order to maintain feasibility (this does not decrease the total expected virtual value on average as the expected virtual value of these agents is non-negative). This very simple algorithm does not use the structure of the social network in any deep way, and is therefore unable to give better approximations in even very simple social networks consisting of a single edge. In order to leverage knowledge of the network structure, we consider a greedy algorithm that iteratively allocates to influential vertices and their neighbors. Our main result shows that this can be used to obtain an \(1/\epsilon + 1 \approx 0.73\)-approximation to the optimal revenue for any distribution of values.

3. CONCLUSION

This letter suggests a new approach to designing optimal auctions. The approach can be summarized as follows:

1. Design an algorithm \(A\) for the virtual value optimization problem. Let \(E_{x \sim D}[A(x)]\) be the expected value of \(A\) when inputs are drawn according to distribution \(D\). Let \(E_{x \sim D}[\text{OPT}(x)]\) be an algorithm that maximizes this expectation.

2. Calculate the average-case approximation \(\alpha = \min_D [E[A(x)] / E[\text{OPT}(x)]]\) where the minimization is over distributions that have non-negative expectation.

Then the resulting auction is an \(\alpha\)-approximation to the optimal auction.

We demonstrated an application of this technique to selling a good with network externalities. In general, we believe our technique is of great use in non-downward closed settings.
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Carbon Footprint Optimization: Game Theoretic Problems and Solutions

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We discuss four problems that we have identified under the umbrella of carbon economics problems: carbon credit allocation (CCA), carbon credit buying (CCB), carbon credit selling (CCS), and carbon credit exchange (CCE). Because of the strategic nature of the players involved in these problems, game theory and mechanism design provides a natural way of formulating and solving these problems. We then focus on a particular CCA problem, the carbon emission reduction problem, where the countries or global industries are trying to reduce their carbon footprint at minimum cost. We briefly describe solutions to the above problem.

Categories and Subject Descriptors: J.4 [Social and Behavioral Sciences]: Economics
General Terms: Carbon Emission, Economics, Game Theory
Additional Key Words and Phrases: Mechanism Design, Auctions, Carbon Footprint Optimization, Carbon Economics

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1. INTRODUCTION

World-wide, there are intense activities by all countries and global organizations to address the issues raised by climate change and global warming. A significant cause for climate change and global warming has been the green house gas (GHG) emissions and other pollutants by industries across the globe. A major contributor among GHGs is the emission of carbon dioxide and hence GHG emissions are also referred as carbon emissions. Carbon emissions are measured in terms of carbon credits where one carbon credit is equal to one ton of carbon dioxide ($CO_2$) emitted. Standard conversion units for other green house gases are available to obtain equivalent $CO_2$ emissions. The well known Kyoto protocol introduced the carbon trading mechanism to be used by global industries or organizations as a means to incentivize them for their emission reduction efforts. The basic approach for carbon trading involves the cap and trade mechanism. A cap and trade system is a market based approach to control pollution that allows global industries or national governments to trade surplus emission allowances after meeting the cap or limit, on those emissions. This mechanism involves two parties, (1) the governing body (or the regulatory authority) and (2) the regulated companies or organizations emitting pollution. The governing body sets a limit called the carbon cap on the total amount of $CO_2$ and other green house gases (equated in terms of $CO_2$) that could be emitted in a given period and will issue rights, or allowances, corresponding to that prescribed level of emissions. Companies that can more efficiently reduce carbon emissions can sell permits to companies that cannot easily afford to reduce emissions. The companies that sell the permits are rewarded while those that purchase permits pay for their negative impact.

1.1 Carbon Economics Problems

We have identified the following four generic problems of carbon emission management, in the context of a country or global industry or organization [Arava et al. 2010]. In the rest of the paper, we use the word agent to represent the country, industry, or organization, as the case may be.

— Carbon Credit Allocation (CCA) Problem: Under the cap and trade mechanism, the allocation of cap to agents becomes an important problem so as to limit the carbon emissions to be less than or equal to the cap. The allocations should consider aspects of varying cost of reductions for different agents, capacity of reduction of each agent, and policy issues. We discuss this problem in Section 2.

— Carbon Credit Buying (CCB) Problem: Agents who cannot reduce their carbon emissions to the level of cap can offset their carbon emissions by buying the required amount of carbon credits from global carbon market or so called carbon exchanges. Agents also have an option to invest in a Clean development mechanism (CDM) and Joint Implementation (JI) projects defined under Kyoto protocol. This gives rise to an interesting problem where the company has to first optimize internally and then buy the extra credits from the market keeping the procurement cost minimum.

— Carbon Credit Selling (CCS) Problem: The agents can earn revenue by selling their surplus carbon credits, to agents that fail to meet the cap. Thus, businesses that are involved in reducing carbon emissions producing low emissions...
can benefit by selling carbon credits in the market. This gives rise to a problem where companies have to optimally make decisions on investments so as to be on the surplus.

—Carbon Credit Exchange (CCE) Problem: The CCB/CCS problem only considers situations where only buyers/sellers are interested in buying/selling carbon credits. An exchange would allow multiple buyers and sellers to trade carbon credits. This gives rise to a large set of problems, similar to that of a stock exchange.

The agents involved in the above problems are typically independent companies or independent units of a company. These agents hold private information such as cost of reducing emissions, capacity of emission reduction, etc and there may not be any incentive for them to report this information truthfully. The four carbon economics problems mentioned above are therefore decision or optimization problems with incomplete information, involving strategic agents. It is required to implement a system-wide solution that satisfies desirable properties such as truthful reporting of private information, efficiency of allocation, budget balance, and voluntary participation. Clearly, a natural way of modeling and solving these problems would be through mechanism design [Narahari et al. 2009; Arava et al. 2010]. To explain this further, in the next section, we will explore the CCA problem in more detail.

2. CARBON CREDIT ALLOCATION PROBLEM

Consider a global industry that has multiple divisions. Each division is an independent unit of the company or a supply chain partner and has capability to measure its carbon emissions truthfully. We assume that the industry under consideration has received a cap on its total emissions from a regulatory authority (for example, the federal government). Let \( E \) be the current (or historical) total number of carbon units emitted by the industry and the cap prescribed is \( C \) units and usually we have \( C < E \). Hence the industry has to reduce or offset \( M = E - C \) emission units.

The industry wishes to achieve this by optimally allocating these \( M \) reduction units among its divisions. As the cost of reductions vary for different divisions, a natural objective of the allocation would be to keep the cost minimum.

The industry here plays the role of a social planner and asks each division to report its cost functions (or cost curves) for the reductions. We assume that the divisions have a finite set of solutions say \( S = \{s_1, s_2, ..., s_m\} \). The cost for implementing the solutions and the respective number of carbon credit reductions obtained is given by the sets \( C = \{cs_1, cs_2, ..., cs_m\} \) and \( R = \{r_1, r_2, ..., r_m\} \). The solutions for carbon emission reductions can be of varying types and may use either consumable items or a new process. If a solution makes use of consumable items, it means that the currently used raw material is replaced by another raw material that is more environment friendly but is perhaps more expensive than the original material. Here we will have: if \( r_i < r_j \), then \( cs_i < cs_j \) \( \forall i, j \in \{1, 2, ..., m\} \). If the set \( C \) is sorted in increasing order, then the set \( R \) will also be in increasing order. For consumable items, the cost can reduce with reductions if the regulatory authority provides an attractive subsidy on the environment friendly materials.

In the case of carbon reduction solutions using a new process, it is reasonable to assume that the solutions are to be implemented in the order given in the set and we have \( \forall i, j \in \{1, 2, ..., m\} \) and \( s_i, s_j \in S \) and \( i < j \), then \( r_i < r_j \) and \( cs_i < cs_j \).
Here we will also have the set $C_R = \{ \frac{c_{s_1}}{r_1}, \frac{c_{s_2}}{r_2}, \ldots, \frac{c_{s_m}}{r_m} \}$ to be an increasing set, where $C_R$ is the set for cost per unit of reduction. Also, if we apply $s_i$ and $s_j$ in order, then the total reduction by combined solution will be given by $r_{ij} = K(r_i + r_j)$ where $r_{ij}$ is the total reduction obtained and $K \geq 1$ is a constant factor. Here the cost curve will always be an increasing curve.

In some cases, the cost curve may become constant after a certain amount of emissions are reduced. We assume that every division has a certain finite reduction capacity (limit on the amount of emission reductions that is possible).

Under the above described settings, the social planner is faced with two kinds of situations:

— **Honest**: Here the individual divisions report their true cost curves. We can formulate emission reduction allocation problem as an optimization problem where the objective is to minimize the cost of reducing $M$.

— **Strategic**: Here the divisions behave strategically and would report their true cost curves only if it is a best response for them. In this case, the social planner has to solve the problem in two steps: (1) elicit the true cost curves and (2) determine an optimal allocation to minimize the cost of reductions.

### 3. OUR CONTRIBUTIONS AND RESULTS

We have proposed algorithms/mechanisms for the emission reduction allocation problem under both the settings described above. In the *honest* case, the problem turns out to be an interesting variant of the knapsack problem and two variants of the problems have been considered: (a) with limited budget and (b) with unlimited budget [Arava et al. 2010]. In both cases (a) and (b), we have used a greedy algorithm which uses the cost curves (bids) of each division and computes the allocation vector which is shown to be optimal. The proposed algorithms can be used by companies to make their decision in budget planning, in deciding how much to invest to meet the immediate cap, how much to invest for future planning, etc.

In [Bagchi et al. 2012], we considered the *strategic* version and proposed a mechanism that a global company may use in allocating emission reductions to its different divisions and supply chain partners towards achieving a required target in its carbon footprint reduction program. The proposed mechanism is strategy-proof and allocatively efficient and uses redistribution mechanisms. We have proposed two kinds of redistribution mechanisms. The first one is based on a reverse auction where the company procures carbon reductions from divisions and then redistribution is done appropriately to reduce budget imbalance. The second one is based on an ingenious forward auction where the company sells permits to avoid emission reductions and redistribution tries to reduce the budget imbalance. We have shown that the forward auction based approach usually outperforms the reverse auction based approach although the reverse auction performs better in some settings.

### 4. CONCLUSION

Carbon credits have become highly valuable and strategic instruments of finance in the global market and it is critical for businesses to have a well thought out strategy for carbon footprint optimization to maximize the global good of the industry. Here we have described one important problem (emission reduction allocation problem).
Other immediate problems that can be formulated and solved are the carbon credit selling, carbon credit buying, and the carbon credit exchange problems. These are problems that exist at the level of an industry as well as the country or world level. We have realized that game theory and mechanism design offer an extremely promising mathematical framework for addressing various carbon economics problems.

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Solution to Exchanges 10.2 Puzzle: Borrowing in the Limit as our Nerdiness Goes to Infinity

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This is a solution to the editor’s puzzle from issue 10.2 of SIGecom Exchanges [Reeves 2011]. The Puzzle asks to determine a point in time such that a lump sum payment of $S will be equivalent to a continuous stream of infinitesimal payments totaling $S, spread evenly over time. The full puzzle can be found online at: http://www.sigecom.org/exchanges/volume_10_2/puzzle.pdf.

Categories and Subject Descriptors: J.4 [SOCIAL AND BEHAVIORAL SCIENCES]: Economics

General Terms: Economics

Additional Key Words and Phrases: Interest

The puzzle asks to determine a point in time such that a lump sum payment of $S will be equivalent to a continuous stream of infinitesimal payments totaling $S, spread evenly over an amount of time T. This note presents a short solution and some of my intuitions.\footnote{The author thanks Daniel Reeves and Assaf Romm for helpful comments and suggestions.} In the spirit of the puzzle, I start with an illustration of the nature of the annualized interest rate.

Let \( r \) be the annualized interest rate. If the rate of interest is fixed, the semi-annual rate \( r_2 \) is such that \( (1 + r_2)^2 = 1 + r \). If I save a dollar for a year, I will then earn \( r_2 \) dollars after the first half of the year. I will earn more money in the second half of the year, because I will begin the period with a higher balance. This is the miracle of compound interest. For a general \( n \), the rate of interest for \( \frac{1}{n} \) of a year \( r_n \), is such that \( (1 + r_n)^n = 1 + r \), hence:

\[
n \log(1 + r_n) = \log(1 + r)
\]

Let \( \tilde{r} := \log(1 + r) \). One can observe easily that the interest rate for time \( \frac{1}{n} \) is \( e^{\tilde{r} \frac{1}{n}} - 1 \), and a continuity argument (or a different partition to sub-periods) implies that the interest rate until \( t \) is \( e^{\tilde{r} t} - 1 \). I shall therefore refer to \( \tilde{r} \) as the instantaneous rate of interest.

I now turn to solve the puzzle, for the case \( S=1 \), and \( T=1 \) year. A continuous (flow) payment of $1 per year discounted to current value is worth:

\[
\int_0^1 e^{-\tilde{r} t} dt = \frac{1 - e^{-\tilde{r}}}{\tilde{r}}
\]  

\( \text{(1)} \)

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The discounted value of a one time payment of $1 at time t is $e^{-\tilde{r}t}$. Specifically, for \( t = 0 \) this expression equals 1, and for \( t = 1 \) it is $e^{-\tilde{r}} = \frac{1}{1+\tilde{r}}$. Hence, for the case of $1 over one year, the solution for the puzzle is the solution of $e^{-\tilde{r}t} = \frac{1}{1+\tilde{r}} \Rightarrow t = -\frac{1}{\tilde{r}} \log \frac{e^{-\tilde{r}}}{1+\tilde{r}}$. Solving for different time periods or sums of money is just a re-parametrization of the current problem.\(^2\) An alternative proof uses the limit of the discrete process described in the puzzle. First, calculate the current value of \( n \) equally spread payments of $1:

\[
\frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{(1+r)^j} = \frac{1}{n} \frac{1 - \frac{1}{1+r}^n}{1 - \frac{1}{1+r}} = \frac{r}{1 + r} \frac{1}{1 - \frac{1}{1+r}^n} \quad \text{as} \quad n \to \infty \quad \frac{1}{1+r} \log(\frac{1}{1+\tilde{r}}) \quad (2)
\]

A payment of 1 at \( t \) has the current value $\frac{1}{(1+r)^t}$. So we are looking for a solution for:

\[
\frac{1}{(1+r)^t} = \frac{r}{1 + r} \frac{1}{1+r} \log(\frac{1}{1+\tilde{r}}) \quad (3)
\]

Rearranging 3 yields:

\[
(1 - t) \log(1 + r) = \log \left( \frac{-r}{\log(1+\tilde{r})} \right) \quad (4)
\]

Recall that $e^{-\tilde{r}} = \frac{1}{1+\tilde{r}} \Rightarrow -\tilde{r} = \log \frac{1}{1+\tilde{r}}$, $e^{\tilde{r}} - 1 = r$, and $\log(1+r) = \tilde{r}$. Substitution yields:

\[
(1 - t)\tilde{r} = \log \frac{1 - e^{\tilde{r}}}{-\tilde{r}} \Rightarrow 1 - t = \frac{1}{\tilde{r}} \log \frac{1 - e^{\tilde{r}}}{-\tilde{r}} \Rightarrow
\]

\[
t = 1 - \frac{1}{\tilde{r}} \log \frac{1 - e^{\tilde{r}}}{-\tilde{r}} = \frac{1}{\tilde{r}} (\log e^{\tilde{r}} - \log \frac{1 - e^{\tilde{r}}}{-\tilde{r}}) = -\frac{1}{\tilde{r}} \log \frac{1 - e^{\tilde{r}}}{-\tilde{r}e^{\tilde{r}}} = -\frac{1}{r} \log \frac{1 - e^{-\tilde{r}}}{\tilde{r}}
\]

the same solution we got before.

It is interesting to observe that \( \lim_{r \to \infty} t = 0 \) and \( \lim_{r \to 0^+} t = \frac{1}{2} \). That is, when the interest rate is low (and the horizon, T, is short), the payment should be made near the mid-point.\(^3\) When the interest rate is high (and the horizon, T, is long), the effect of compounding is large, and so the fair payment timing is early. For example, when \( T=1 \) year and \( r \) is 10% the fair payment time is close to .496. To see why \( t \) approaches 0 as \( r \) grows to infinity, imagine that \( r \) is very large. After 1% of the period had passed, 1% of the money must have been paid in the continuous payment scheme. With \( r \) large enough, leaving this money in the bank for 1% of the period would yield more than $S. Therefore, the fair \( t \) must be earlier than 2% of the period.

Finally, the puzzle assumed that nerdiness is of the type we are used to - nerds who like complicated calculations. However, one can imagine a world where agents

\(^2\)The solution remains the same for different values of \( S \). For different \( T \), \( r \) should be replaced with the interest over \( T \) years, and the solution should be interpreted as a proportion out of \( T \).

\(^3\)Note that when \( r = 0 \) the timing doesn’t matter, and any \( t \) solves the problem.
prefer to avoid the variation in the timing of the lump sum payoff, and find it difficult to pay continuously. Equation 1 offers a simple solution for such agents. An infinitesimal (flow) payment of $1 would yield the same profit as an interest free loan of \( \frac{1}{r} \) for the same duration. Moreover, the solution suggests a simple compensation scheme for the provision of continuous services; \( \frac{S}{r} \) are deposited in an escrow account and kept in the account for the duration. Then, the interest accrued is paid to the service provider and the principal is refunded to the client.

REFERENCES

Reeves, D. 2011. Borrowing in the limit as our nerdiness goes to infinity. SIGecom Exch. 10, 2 (June), 50–50.