Logit Dynamics: A Model for Bounded Rationality

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We describe logit dynamics, which are used to model bounded rationality in games, and their related equilibrium concept, the logit equilibrium. We also present some results about the convergence time of these dynamics and introduce a suitable approximation of the logit equilibrium. We conclude by describing some interesting future extensions to logit dynamics.

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1. INTRODUCTION

Classical Game Theory assumes that agents have complete knowledge of the game (they know all the players, their set of strategies and their utility functions and they know that other players know, and they know that others know that they know, etc.). It also assumes the players have unlimited computational power to select a strategy that maximizes their utility given the strategies played by other players. However, in many cases players' decisions can be influenced by limited knowledge and limited computational capabilities. Thus, to have a more precise description of phenomena emerging in these settings we need a model that can capture the behavior of agents with *bounded rationality* and that may sometimes make wrong decisions.

An example of such a model is the *logit update rule* [McFadden 1974]. According to this rule, players update their strategies with respect to a parameter β (that represents the level of rationality or knowledge) and the state of the system (i.e., the strategies currently played by the players). Roughly speaking, the logit update rule can be seen as a noisy version of the classical best response update rule, where β is the bias towards choices that are good. A small β represents the situation where players are subject to strong noise or they have very limited knowledge of the game and, therefore, choose their strategies "nearly at random"; a large β, instead, represents the situation where players "almost surely" play the best response.

Blume [1993] introduced the logit update rule in game dynamics through logit dynamics. We will describe these dynamics and discuss some of their properties.

2. LOGIT DYNAMICS AND LOGIT EQUILIBRIA

Consider a strategic game $\mathcal{G} = ([n], S_1, \ldots, S_n, u_1, \ldots, u_n)$, where $[n] = \{1, \ldots, n\}$ is a finite set of players, S_i is the finite set of strategies for player $i \in [n]$, $S = S_1 \times \cdots \times S_n$ is the set of strategy profiles and $u_i \colon S \to \mathbb{R}$ is the utility function of player $i \in [n]$. The *logit dynamics* for a game $\mathcal{G}$ proceed as follows: at each time

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step (i) Select one player $i \in [n]$ uniformly at random; (ii) Update the strategy of player i according to the logit update rule with parameter $\beta > 0$ over the set S_i of her strategies. That is, the dynamics select a strategy $s \in S_i$ with probability $\sigma_i(s \mid \mathbf{x}) = e^{\beta u_i(s,\mathbf{x}_{-i})}/Z_i(\mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_n) \in S$ is the current strategy profile and $Z_i(\mathbf{x}) = \sum_{z \in S_i} e^{\beta u_i(z,\mathbf{x}_{-i})}$ is the normalizing factor.

These dynamics have been extensively adopted in Economics and, recently, in Computer Science to model the spread of innovation in social networks [Ellison 1993; Young 2000; Montanari and Saberi 2009]. These works focused on the time the dynamics take for hitting a specific pure Nash equilibrium of the game. However, Nash equilibria may not be an adequate solution concept for these dynamics. Indeed, there is always a chance, which is inversely proportional to the rationality level β, that players deviate from these strategy profiles.

To address this issue, Auletta et al. [2010] introduce a new equilibrium concept for logit dynamics, named *logit equilibrium*, describing the long-run behavior of the system (which states appear more frequently in the long run) and defined by a probability distribution over the pure strategy profiles of the game. In fact, it is easy to see that the logit dynamics for a game $\mathcal{G}$ define a Markov chain with the set S of strategy profiles as state space. Then, the logit equilibrium is defined as the stationary distribution of this Markov chain, i.e. the distribution π such that $\pi P = \pi$, where P is the transition matrix of the Markov chain. It is not hard to see that the chain defined by the logit dynamics is ergodic and, hence, any strategic game possesses a logit equilibrium and this is unique. We note that the absence of either of these guarantees is often considered a weakness of pure Nash equilibria.

If the time the dynamics take to reach an equilibrium is long, then the system spends most of its life outside of the equilibrium and thus the relevance of this concept is almost completely lost. For this reason, it is important to bound the time that the dynamics takes to reach either the equilibrium or some suitable approximations of it. The next section provides some results in this direction.

3. CONVERGENCE TIME AND METASTABLE DISTRIBUTIONS

Auletta et al. [2012a] give general bounds on the convergence time of logit dynamics for wide classes of games. These results show that there are games for which the convergence time is bounded from above by a polynomial in the number of players, and by an exponential function in the rationality level and in some structural properties of the game. Thus, as β increases (i.e., players become more rational), the dynamics take longer to reach the equilibrium. Indeed, for high β players tend to play their best response and the system is likely to remain in a pure Nash equilibrium (if any) for a long time. This behavior slows down the convergence to the logit equilibrium, whenever the stationary distribution assigns high probability to other profiles (e.g., this is the case if there are other similar Nash equilibria). Auletta et al. [2012a] also show games for which the convergence time is bounded by a function independent of β. Unfortunately, this function can be exponential in the number of players.

Since the convergence time of the dynamics can be large, it becomes interesting to understand what happens during the transient phase of the dynamics. Is this phase completely chaotic, or can we still spot some regularities? Are we able to describe the behavior of the system even before stationarity has been reached? Is it possible that on a timescale shorter than the convergence time the chain is "metastable", i.e., it stays close to some subset of the state space, while in a timescale comparable to the convergence time it jumps from one metastable configuration to another?

In order to answer these questions, Auletta et al. [2012] introduce the definition of *metastable distribution*. Roughly speaking, a distribution μ is (ε, T)-metastable for the dynamics if, selecting the starting profile according to μ, the dynamics stays at distance at most ε from μ for at least T steps. Metastable distributions can be seen as approximations of the stationary distribution. Indeed, they remain stable for a time which is long enough for an observer (in computer science terms, we assume this time is super-polynomial), while the stationary distribution remains stable forever. The hope is that whenever there is a starting profile from which the convergence to the logit equilibrium takes too long, then, from that profile, the dynamics should converge quickly to a metastable distribution. Ferraioli and Ventre [2012] take the first steps in this direction.

4. FUTURE DIRECTIONS

Logit dynamics assume that only one player updates her strategy at any time. It would be interesting then to model also players concurrently updating their strategies. Auletta et al. [2012b] give preliminary results in this direction.

Another implicit assumption of logit dynamics is that all players have the same rationality level. However, it would be interesting to extend the analysis of logit dynamics by taking into account that different players may have different levels of rationality depending on different personal attitudes.

Last but not least, it would be extremely interesting to understand in which way we can influence the evolution of a game whose agents are not fully rational, in order to push the system towards desired directions.

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