The Price of Stability of Fair Undirected Broadcast Games is Constant

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Bounding the price of stability of undirected network design games with fair cost allocation is a challenging open problem in the Algorithmic Game Theory research agenda. Even though the generalization of such games in directed networks is well understood in terms of the price of stability (it is exactly H_n, the n-th harmonic number, for games with n players), far less is known for network design games in undirected networks. The upper bound carries over to this case as well, while the best known lower bound is 2.245. For more restricted but interesting variants of such games, such as broadcast and multicast games, sublogarithmic upper bounds are known, while the best known lower bounds are 1.818 and 1.862, respectively. In this letter, we discuss a recent breakthrough in this field of research: an $O(1)$ upper bound on the price of stability for undirected broadcast games.

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1. PROBLEM STATEMENT AND RELATED WORKS

A *network design game with fair cost allocation* (from now on, simply, *network design game*) is defined by an edge weighted connected graph $G = (V, E, c)$, with $c : E \to \mathbb{R}_{\geq 0}$, and a set N of n players, where each player $i \in N$ wants to connect a pair (s_i, t_i) of nodes of G. Intuitively, nodes of G model network endpoints and each edge $e \in E$ represents a link of realization cost $c(e)$ which can be potentially established among two endpoints. Each player $i \in N$ wants to choose an (s_i, t_i)-path in G (a *strategy*) so as to minimize her cost, knowing that the cost of each edge in a *strategy profile* (a combination of strategies, one for each player) is equally shared among the players making use of it. Depending on whether G is directed or undirected, we talk of directed or undirected network design games, respectively. A

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relevant and extensively investigated special case of network design games occurs when $s_i = s$ for each $i \in N$ and is called *multicast game*; moreover, *broadcast games* are multicast games in which there is a player associated with every node of G except for s.

Each network design game is an instance of congestion games, a well-known and studied class of games admitting an *exact potential function*, that is, a function from the set of all possible strategy profiles to the reals such that the difference in the potential value between two strategy profiles that differ in the strategic choice of a single player is equal to the difference of the costs experienced by this player in these two profiles, see [Rosenthal 1973]. This immediately implies the existence of a *pure Nash equilibrium* for these games: in fact, since any sequence of improving deviations by the players strictly decreases the value of the potential, a local minimum of the potential, which corresponds to a pure Nash equilibrium, will eventually be reached in a finite number of steps.

The quality of a strategy profile is naturally measured by the realization cost of the induced network, which is given by the sum of the realization costs of the selected edges. Unfortunately, a network corresponding to a pure Nash equilibrium may be much more costly than the ones which could be potentially enforced by a dictatorial designer. To this end, the notions of *price of anarchy* and *price of stability* measure the deterioration of performance due to the presence of selfish players. In particular, the price of anarchy is defined as the worst-case ratio of the realization cost of the network induced by a pure Nash equilibrium and that of a minimum cost network which satisfies the connectivity requirements of all the players. The price of stability, instead, considers the best-case ratio, thus bounding the minimum loss of performance that has to be suffered because of the lack of a central coordinator.

Network design games were introduced in [Anshelevich et al. 2004], where it is proved that the price of stability is at most H_n (it is quite easy to see that the price of anarchy is equal to n even for undirected broadcast games). The proof considers a Nash equilibrium that can be reached from a minimum cost network when the players perform arbitrary improving deviations. The main used argument is that the potential of this Nash equilibrium is strictly smaller than the one of the minimum cost network and the proof follows by exploiting the fact that the potential function of Rosenthal approximates the cost of the network induced by a strategy profile within an H_n multiplicative factor. For directed graphs, the H_n bound was also proved to be tight even in broadcast games. Although the upper bound proof carries over to undirected network design games, the lower bound does not. H_n is the only known upper bound for undirected network design games, while better results have been achieved for undirected multicast and broadcast games, for which [Li 2009] and [Fiat et al. 2006] present upper bounds of $O(\log n/ \log\log n)$ and $O(\log\log n)$, respectively. Recently, the latter has been improved to $O(\log\log\log n)$ in [Lee and Ligett 2013]. Despite these super-constant upper bounds and the effort of many researchers, the best known lower bounds for these three classes of games are 2.245, 1.862 and 1.818, respectively, as shown in [Bilò et al. 2013a].

2. OUR CONTRIBUTION

In our recent paper [Bilò et al. 2013b], we show the following breakthrough result.

THEOREM 2.1. *The price of stability of undirected broadcast games is $O(1)$.*

The theorem is proved in a constructive way by giving an (inefficient) algorithm which, starting from an optimal strategy profile (for undirected broadcast games such a profile is induced by a minimum spanning tree T^* of G), carefully combines improving deviations with batches of deviations which, although not being improving deviations for any player, are able to lower the potential value when considered as a whole. When the algorithm terminates it returns a Nash equilibrium whose induced network has a realization cost proportional to the one of T^*.

The key ingredient of our approach is a novel technique that we call *homogenization*. It is based on the idea that, when two players i and j at distance d_{ij} from each other in T^* experience a significantly different cost in a certain profile S, then it is possible to perform a *homogenization process*, that is a "transformation" of S into a new homogeneous profile S' with a lower potential. A homogeneous profile has the fundamental property that the difference between the costs incurred in it by any two players i and j is upper bounded by a poly-logarithmic factor in d_{ij}.

Assume that a player i, by performing an improving deviation, introduces an edge e not belonging to T^* in the current strategy profile. Such an edge is called a *critical edge*. It is possible to show that, after homogenizing, either there exists a subsequent improving deviation that immediately removes e from the current profile or there exists another transformation, called the *absorption process*, which rebuilds a significant portion of T^* around edge e and yet decreases the potential value. In particular, thanks to the fundamental property of homogeneous profiles, the absorption process removes all critical edges (if any) previously inserted by all the players whose distance from i in T^* is exponential in the realization cost of e.

By suitably scheduling improving deviations together with homogenization and absorption processes, our algorithm finally arrives at a local minimum S of the potential function (i.e., a Nash equilibrium) approximating the realization cost of T^* by a constant factor.

Many technicalities need to be addressed in order to make this general idea work. For instance, the fact that critical edges have to be partitioned into classes with respect to their realization cost, the fact that during the homogenization and absorption processes the costs of the involved players might decrease due to the effect of the congestions caused by previous deviating players, and so on. Details can be found in [Bilò et al. 2013b].

3. OPEN PROBLEMS

Although asymptotically matching upper and lower bounds have been achieved for the price of stability of undirected broadcast games, there is still a huge gap in the constants hidden inside the big–O notation which need to be further reduced. Moreover, and perhaps more importantly, it would be nice to understand whether homogenization could be exploited to improve the currently known upper bounds also for the more general cases of undirected multicast games and undirected network design games.

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