Algorithms as Mechanisms: The Price of Anarchy of Relax-and-Round

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We show that algorithms that follow the relax-and-round paradigm translate approximation guarantees into Price of Anarchy guarantees, provided that the rounding is oblivious and the relaxation is smooth. We use this meta result to obtain simple, near-optimal mechanisms for a broad range of optimization problems such as combinatorial auctions, the maximum traveling salesman problem, and packing integer programs. In each case the resulting mechanism matches or beats the performance guarantees of known mechanisms.

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1. INTRODUCTION

The "classic" approach to mechanism design is to devise mechanisms that incentivize truthful reporting. For settings where the private information of each agent consists of a single number this boils down to finding allocation rules that are monotone in each agent's private value. Many natural approximation algorithms either satisfy this monotonicity constraint or can be tweaked to satisfy it.

For more general settings, where the private information is multi-dimensional, designing truthful mechanisms is an intriguing task, and no general transformation for turning approximation algorithms into truthful mechanisms with comparable performance is known. The most general positive algorithmic results in this context are so-called black-box reductions. These turn algorithms of a certain class into truthful mechanisms by making polynomially many calls to the approximation algorithm. Examples range from the early work of Lavi and Swamy [2005] to the more recent work of Dughmi and Roughgarden [2014].

We consider here an alternative approach that we refer to as "algorithms as mechanisms". This approach is different from the black-box reduction approach in that it takes an approximation algorithm as is and couples it with a simple payment rule, such as pay-your-bid. Of course, this will typically not lead to a mechanism where participants want to report their true values. Instead we seek to understand

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which approximation algorithms guarantee that all equilibria or even all learning outcomes of the resulting mechanism are close to optimal.

In fact, in a landmark result Lucier and Borodin [2010] showed that greedy algorithms have this property: Any equilibrium of a greedy algorithm that is an α-approximation algorithm is within $O(\alpha)$ of the optimal solution.

Our main result is to show that the common design principle of *relaxation and rounding* also approximately preserves the approximation guarantee as a Price of Anarchy guarantee provided that the relaxation is *smooth* and the rounding process is *oblivious* (more on this below). The canonical example of this approach are integer linear programs that are relaxed to a fractional domain, then the relaxation is solved to optimality and converted into an integer solution via randomized rounding. Other examples that follow this pattern come from semi-definite programming or involve relaxing one combinatorial problem to another.

Our meta result has—as we show—far-reaching consequences in mechanism design: It leads to novel, simple, yet near-optimal mechanisms for sparse packing integer programs such as multi-unit auctions and generalized matching, for the maximum traveling salesman problem, for combinatorial auctions and for single source routing problems. In all cases we obtain Price of Anarchy bounds that match or beat known Price of Anarchy guarantees, or they are the first non-trivial guarantees for the respective problem.

2. MAIN RESULT

Our main result concerns the algorithmic blueprint of *relaxation and rounding* (see, e.g., [Vazirani 2001]). In this approach a problem Π is *relaxed* to a problem Π', with the purpose of rendering exact optimization computationally tractable. Having found the optimal relaxed solution x', another algorithm derives a solution x to the original problem. This process is called *rounding*.

A (possibly randomized) rounding scheme r for translating a solution x' to the relaxed problem Π' into a solution $x = r(x')$ to the original problem Π is *α-approximate oblivious*, where $\alpha \geq 1$, if the rounding depends on the solution x' only and for all possible valuation profiles each agent is guaranteed to get, in expectation, a $1/\alpha$-fraction of the value that it would have had for the solution to the relaxed problem.

Clearly an α-approximate oblivious rounding scheme, when combined with optimally solving the relaxed problem, leads to an approximation ratio of α. We show that it also approximately preserves the Price of Anarchy of the relaxation assuming the relaxation is smooth. We focus on pay-your-bid mechanisms for concreteness. Our result actually applies to a broad range of mechanisms and can also be extended to include settings where the relaxation is not solved optimally.

All our bounds go through smoothness [Roughgarden 2009; 2012; Syrgkanis and Tardos 2013] and therefore apply to a broad range of equilibrium concepts.

THEOREM 2.1. *If the pay-your-bid mechanism M for problem Π is obtained by solving the relaxation Π' optimally and applying an α-approximate oblivious rounding scheme and the pay-your-bid mechanism M' for problem Π' that solves the relaxation optimally has a Price of Anarchy of β via smoothness, then the mechanism M has a Price of Anarchy of $2\alpha\beta$ via smoothness.*

In most applications, including the ones below, the factor 2 loss from the general framework can be avoided.

3. APPLICATIONS

Our result significantly broadens the algorithmic toolbox for designing mechanisms with near-optimal equilibria. We use the richer set of tools to obtain novel mechanisms for a broad range of optimization problems. We note that in all of our applications, it is important to use the relaxation to show smoothness of the problem. For example, optimally solving the original (integer) problem would give a very high Price of Anarchy.

(1) Combinatorial Auctions. We use the ingenious rounding scheme by Feige [2009] to obtain a Price of Anarchy of $4e/(e-1)$ for fractionally subadditive valuations. For the more general class of MPH-k valuations defined in [Feige et al. 2014] we obtain a Price of Anarchy of $O(k^2)$.

(2) Sparse Packing Integer Programs. Here we use the rounding scheme by Bansal et al. [2010] to show a Price of Anarchy of $O(d^2)$ when the column sparsity is d. This result applies to multi-unit auctions with general valuations or the generalized assigment problem (where $d = 1$) or combinatorial auctions with bounded bundle size (where d is the maximum bundle size).

(3) Maximum Traveling Salesman Problem. We show that the classic algorithm by Fisher et al. [1979] that relaxes the problem to finding a cycle cover yields a Price of Anarchy of 12. We also show how this bound can be improved to 9 by using the algorithm of Paluch et al. [2012].

(4) Single-Source Unsplittable Flow. We adapt the "original" randomized rounding algorithm of Raghavan and Thompson [1987; 1988] to obtain a Price of Anarchy of $2(1 + \epsilon)$ for high enough capacities.

4. FUTURE WORK

It is still largely open whether reductions similar to the one presented here also apply to other classes of algorithms or whether greedy and relax-and-round algorithms are the only classes that admit such a result. More generally, one could try to charaterize which algorithms lead to mechanisms with small Price of Anarchy. A first step towards such a characterization was recently made by Dütting and Kesselheim [2015].

REFERENCES

Bansal, N., Korula, N., Nagarajan, V., and Srinivasan, A. 2010. On k-column sparse packing programs. In *Proc. of the 14th Conference on Integer Programming and Combinatorial Optimization*. 369–382.

Dughmi, S. and Roughgarden, T. 2014. Black-box randomized reductions in algorithmic mechanism design. *SIAM Journal on Computing 43,* 1, 312–336.

Dütting, P. and Kesselheim, T. 2015. Algorithms against anarchy: Understanding non-truthful mechanisms. In *Proc. of the 16th Conference on Economics and Computation*. 239–255.

Feige, U. 2009. On maximizing welfare when utility functions are subadditive. *SIAM Jornal on Computing 39,* 1, 122–142.

Feige, U., Feldman, M., Immorlica, N., Izsak, R., Lucier, B., and Syrgkanis, V. 2014. A unifying hierarchy of valuations with complements and substitutes. *Electronic Colloquium on Computational Complexity 21*, 103.

Fisher, M. L., Nemhauser, G. L., and Wolsey, L. A. 1979. An analysis of approximations for finding a maximum weight hamiltonian circuit. *Operations Research 27,* 4, 799–809.

Lavi, R. and Swamy, C. 2005. Truthful and near-optimal mechanism design via linear programming. In *Proc. of the 46th Symposium on Foundations of Computer Science*. 595–604.

Lucier, B. and Borodin, A. 2010. Price of anarchy for greedy auctions. In *Proc. of the 21st Symposium on Discrete Algorithms*. 537–553.

Paluch, K. E., Elbassioni, K. M., and van Zuylen, A. 2012. Simpler approximation of the maximum asymmetric traveling salesman problem. In *Proc. of the 29th International Symposium on Theoretical Aspects of Computer Science*. 501–506.

Raghavan, P. 1988. Probabilistic construction of deterministic algorithms: Approximating packing integer programs. *Journal on Computer Systems Sciences 37,* 2, 130–143.

Raghavan, P. and Thompson, C. 1987. Randomized rounding: a technique for provably good algorithms and algorithmic proofs. *Combinatorica 7*, 365–374.

Roughgarden, T. 2009. Intrinsic robustness of the price of anarchy. In *Proc. of the 41st Symposium on Theory of Computing*. 513–522.

Roughgarden, T. 2012. The price of anarchy in games of incomplete information. In *Proc. of the 13th Conference on Electronic Commerce*. 862–879.

Syrgkanis, V. and Tardos, É. 2013. Composable and efficient mechanisms. In *Proc. of the 45th Symposium on Theory of Computing*. 211–220.

Vazirani, V. V. 2001. *Approximation Algorithms*. Springer-Verlag New York, Inc., New York, NY, USA.