A Proof of the Nisan-Ronen Conjecture — An Overview

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This note presents an overview of our recent publication, which validates a conjecture proposed by Nisan and Ronen in their seminal paper [Nisan and Ronen 2001]. We show that the optimal approximation ratio for deterministic truthful mechanisms for makespan-minimization by a set of n unrelated machines is n.

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The seminal work [Nisan and Ronen 2001] set the foundations of the field of *algorithmic mechanism design* by probing the computational and information-theoretic limits of *mechanism design*. Mechanism design, a celebrated branch of game theory and microeconomics, studies the design of algorithms (called mechanisms) in environments where the input is privately held and provided by selfish participants. A mechanism for an optimization problem, on top of the traditional algorithmic goal (that assumes knowledge of the input), bears the extra burden of providing incentives to the participants to report their true input. One of the main thrusts in this research area is to demarcate the limitations imposed by truthfulness on algorithms. To what extent are mechanisms less powerful than traditional algorithms?

The objective of the scheduling problem is to minimize the makespan of allocating m tasks to n unrelated machines, where each machine i needs $t_{i,j}$ units of time to process task j. The problem combines various interesting properties. First, it belongs to the most challenging and least explored area of *multi-dimensional* mechanism design, as the private information is multi-dimensional (i.e., for player i, the private values $(t_{i,j})_{j=1}^{m}$ are a vector). In contrast, the *related* machines scheduling belongs to *single-dimensional* mechanism design, which is well-understood, and for which the power of truthful mechanisms does not substantially differ from the best non-truthful algorithms: not only can truthful mechanisms compute exactly opti-

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mal solutions (if one disregards computational issues [Archer and Tardos 2001]), but a truthful PTAS exists [Christodoulou and Kovács 2013]. Second, the objective of the scheduling problem has a min-max objective, which from the mechanism design point of view is much more challenging than the min-sum objective achievable by the famous VCG [Vickrey 1961; Clarke 1971; Groves 1973] mechanism. VCG is truthful and can be applied to the scheduling problem, but it achieves a very poor approximation ratio, equal to the number of machines n [Nisan and Ronen 2001].

Is there a better mechanism for scheduling than VCG? Nisan and Ronen [Nisan and Ronen 2001] conjectured that the answer should be negative, but for the past two decades, the question has remained open. In our work [Christodoulou et al. 2023] we validate the conjecture.

THEOREM 1. *There is no deterministic truthful mechanism with approximation ratio better than n for the problem of scheduling n unrelated machines.*

Over the years various research attempts with limited success have been made to improve the original lower bound of 2 by Nisan and Ronen. For example, the bound was improved to 2.41 in [Christodoulou et al. 2009], and later to 2.61 in [Koutsoupias and Vidali 2012], which held as the best bound for over a decade. More recently, the lower bound was improved to 2.75 by Giannakopoulos, Hammerl, and Poças [Giannakopoulos et al. 2021], and then to 3 by Dobzinski and Shaulker [Dobzinski and Shaulker 2020]. These improved bounds represented progress in the field, but they left a huge gap between the lower and upper bounds. The first non-constant lower bound for the truthful scheduling problem was given in [Christodoulou et al. 2021a], which showed a lower bound of $\Omega(\sqrt{n})$.

1. THE MAIN ARGUMENT OF THE PROOF

We consider a restricted class of inputs given by a multi-graph where each node is a machine and each edge is a task [Christodoulou et al. 2021b]. For an edge e, we use the notation $e = \{i, j\}$ to denote its vertices i and j, although they do not determine e uniquely. An edge $e = \{i, j\}$ corresponds to a task that has extremely high values for nodes other than i and j, which guarantees that any algorithm with approximation ratio at most n must allocate it to either machine i or machine j (see Figure 1 for an illustration).

The argument deals with multi-cliques with very high *multiplicity*[1], in which *every edge has an endpoint with value 0* (see Figure 3 for an example). The goal is to carefully select a subgraph of this multi-clique and change the values of some of its edges to obtain a lower bound on the approximation ratio. The fact that one of the two values of every edge is 0 is very convenient: a lower bound on the approximation ratio of the subgraph is a lower bound on the approximation ratio of the whole multi-clique as well, since the other edges do not affect the cost of the optimal allocation.

For an instance v, we use the notation v_i^e to denote the value of node i for an edge $e = \{i, j\}$. In most of the argument, we fix the values of the multi-clique and we focus on the boundary functions.

[1]The multiplicity of a multi-graph is defined to be the minimum multiplicity among its edges.

$$\begin{pmatrix} 5 & 3 & 7 & 4 \\ 1 & 2 & \infty & \infty \\ \infty & \infty & 9 & 6 \end{pmatrix}$$

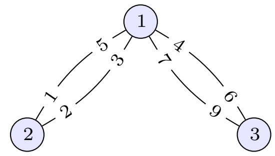

Fig. 1. A multi-star instance with three players and four tasks in matrix form (left) and in graph form (right). The symbol ∞ denotes values that are extremely high compared to the other values. This instance is a multi-star, in which player 1 is the root and players 2 and 3 are the leaves.

Definition 2 *Boundary function.* Fix a mechanism and consider a multi-clique with values v. For an edge $e = \{i, j\}$, the boundary function $\psi^e_{i,j}(z)$ is the threshold value for the allocation of e to node i. More precisely, if we keep all the other values fixed and change the value of e for node j to z, then e is allocated to i if $v^e_i < \psi^e_{i,j}(z)$ and to j if $v^e_i > \psi^e_{i,j}(z)$.

A boundary function $\psi^e_{i,j}(\cdot)$ may depend on the other values of v. Truthfulness severely restricts the class boundary functions. In particular, when we fix all values except the value of a single task, the boundary function $\psi^e_{i,j}(z)$ must be increasing in z. This is single-parameter monotonicity and it is mainly used in the Nice Multi-Star (Theorem 5). A more severe condition on the boundary functions and their relationship comes from multi-parameter truthfulness, that determines how the allocation partitions the space of values (see for example Figure 2). Specifically, the multi-parameter truthfulness for 2 players and 2 tasks plays a central role in the proof and it is repeatedly employed as the main tool for proving the Box Theorem (Theorem 6).

The aim of the proof of the main theorem (Theorem 1) is to show — by the probabilistic method — that there exists a multi-clique of sufficiently high multiplicity that contains *a star with approximation ratio at least* n, when we keep the values of all other edges fixed. In fact, the argument aims to show that the bound on the approximation ratio for the star is arbitrarily close to $n - 1$. The extra $+1$ in the approximation ratio comes, almost for free, by adding a loop to the root of the star, or equivalently an additional edge between the root and another node j with very high value for j.

To show that there exists a star S with approximation ratio $n - 1$, we *roughly* aim to show that there exists a star with some root i, with the following properties:

(1) every edge $e = \{i, j\}$ of S has value 0 for i and *the same* value z for the leaves, for some $z > 0$.

(2) the sum of the values of the boundary functions over all edges $\sum_{e \in S} \psi^e_{i,j}(z)$ is at least $(n - 1)z$.

(3) the mechanism allocates all edges to the root, when we change its values to $\psi^e_{i,j}(z)$ for all $j \neq i$.

It follows immediately that such a star has approximation ratio $n - 1$: the mechanism allocates all tasks to the root with makespan $\sum_{e \in S} \psi^e_{i,j}(z) \geq (n - 1)z$, while a better allocation is to allocate all tasks to the leaves with makespan z.

A star that satisfies the second property will be called *nice* and the third property *box*. In [Christodoulou et al. 2021a], we used an argument that is similar to the

Box Theorem, which establishes the box property for many stars. Actually the Box Theorem is a cleaner and stronger argument than the one in [Christodoulou et al. 2021a], and one can use it to obtain the results of that work directly. To completely resolve the Nisan-Ronen conjecture we needed to work with multi-cliques, not only stars as in [Christodoulou et al. 2021a], and the Nice Multi-Star Theorem allows us to focus on a particular multi-star of the multi-clique.

For technical reasons we need to work with approximate notions of niceness and box-ness.

Definition 3 *Nice star and nice multi-star.* For a given $\varepsilon > 0$ and an instance v, a star S with root i and leaves all the remaining $(n-1)$ nodes is called ε-nice, or simply nice, if there exists $z > 0$ such that:

(1) every edge $e = \{i, j\}$ of S has value $v_i^e = 0$ for root i and $v_j^e \in (z, (1+\varepsilon)z)$ for leaf j

(2)

$$\sum_{e \in S} \psi_{i,j}^e(v_j^e) \geq (1 - 3\varepsilon)(n-1)z. \tag{1}$$

A multi-star is nice if *all of its stars* with $n-1$ leaves are nice, with the *same* z.

By letting ε tend to 0, v_j^e can be arbitrarily close to z. Next we define boxes[2].

Definition 4 *Box.* For a given $\delta > 0$ and instance v, a star S with root i is called δ-box, or simply box, if every edge $e = \{i, j\}$ of S has value 0 for i and the mechanism allocates all edges to i, when we change their value for i to $\psi_{i,j}^e(v_j^e) - \delta$ for every leaf j of S.

Now that we have the definitions of nice multi-stars and box stars, we can state the two main theorems that almost immediately establish the main result. The first theorem establishes the existence of nice multi-stars of arbitrarily high multiplicity (see Figure 3). The second theorem asserts that nice multi-stars with sufficiently high multiplicity contain a box star of $n-1$ leaves (see Figure 3).

Theorem 5 Nice Multi-Star. *For every mechanism with bounded approximation ratio and every q, there exists a multi-clique that contains a nice multi-star with multiplicity q.*

Theorem 6 Box. *Fix $\delta, \varepsilon > 0$ and a mechanism with approximation ratio at most n. Consider an instance that contains a multi-star, of sufficiently high multiplicity, in which all values of the root i are 0 and all values of the leaves are in $(z, (1+\varepsilon)z)$. Then the multi-star contains a star with $n-1$ leaves, which is a δ-box.*

The proof of the main result (Theorem 1) follows immediately from the above two theorems. Use Theorem 5 to find a multi-clique that contains a nice multi-star with sufficiently high multiplicity. Use Theorem 6 to find a nice box inside it. The next lemma makes this precise.

[2]See Figure 2 for an illustration.

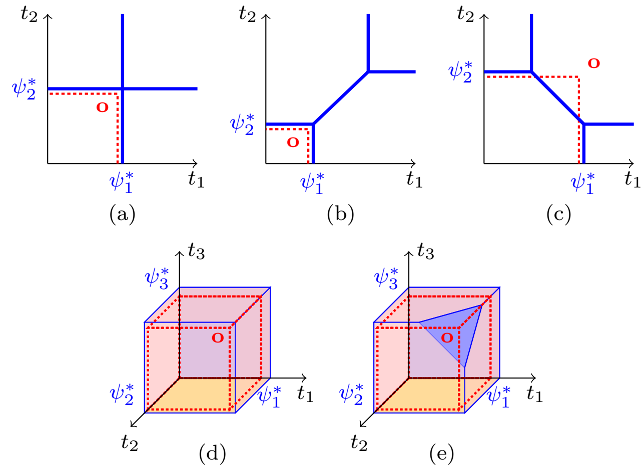

Fig. 2. (Box). Allocation partitions of the root values for a star of 2 leaves (a)-(c) and 3 leaves (d)-(e); in the latter case, only part of the allocation partition is shown. Call the root i and leaves $j \in \{1, 2, 3\}$. If we denote the edges of the star by e_j, the figure uses the shorthand: $t_j = v_i^{e_j}$ for the values of the root, and $\psi_j^* = \psi_{i,j}^{e_j}(v_j^{e_j})$ for the boundary values. Truthfulness restricts the shapes and boundaries of the allocation areas. The dotted red lines correspond to values $\psi_j^* - \delta$ of the box definition. Cases (a), (b), and (d) are boxes, as the corner o is inside the region where the root gets all the tasks. On the other hand, cases (c) and (e) are not boxes, since the corner point o lies outside this region.

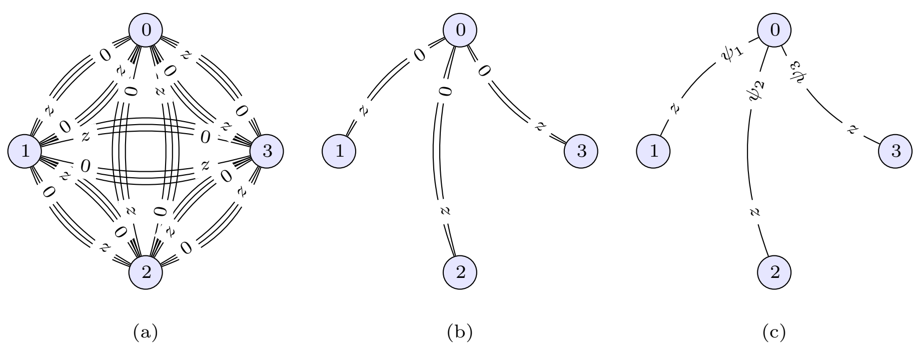

Fig. 3. This is an illustration of the components that appear in the statement of Theorem 1. (a) shows a multi-clique with $n = 4$ nodes and multiplicity 6. It should be noted that the actual multiplicity needed is much higher. For simplification, we use z to denote non-zero values, which are not necessarily the same for all edges. (b) shows a multi-star which is a subgraph of the multi-clique. If this is a *nice* multi-star, the value z is approximately the same for all leaves. (c) shows a simple star of this nice multi-star, selected by the Box Theorem 6. The nice-ness property roughly guarantees that $\psi_1 + \psi_2 + \psi_3 \geq 3z$, and the box-ness property guarantees that all tasks will be allocated to the root. This will give approximation ratio roughly 3; we can increase this to $n = 4$ by adding loops. Note that the remaining edges — which do not appear in (c) — do not contribute to the optimal makespan, because one of their values is 0.

Lemma 7. *A δ-nice box with a loop in the root, in which all values of the root are 0 and all values of the leaves are in $(z,(1+\varepsilon)z)$, has approximation ratio n, as δ and ε tend to 0.*

Proof. Take a nice box and consider the instance when we change the values of the root i to $\psi^e_{i,j}(v^e_j)-\delta$ for all $j\neq i$. By the box-ness property all the edges are allocated to the root. Change now the value of the loop to z and decrease the values of the root to $\psi^e_{i,j}(v^e_j)-2\delta$. The task that corresponds to the loop must still be allocated to root, even when we increase its value to z. By applying monotonicity, the allocation of the edges remains the same. The makespan of the mechanism is

$$z+\sum_{j\neq i}(\psi^e_{i,j}(v^e_j)-2\delta)\geq z+(1-3\varepsilon)(n-1)z-2(n-1)\delta,$$

while the optimal makespan is at most $(1+\varepsilon)z$, (when the root gets the loop and the leaves get the remaining edges). The ratio tends to n as δ and ε tend to 0. □

2. CONCLUSION

Our work [Christodoulou et al. 2023] validates the Nisan-Ronen conjecture, by establishing a lower bound for all deterministic truthful mechanisms. The case of randomized or fractional mechanisms is still open and it appears to be challenging; the best known lower bound of the approximation ratio is 2 [Mu'alem and Schapira 2018; Christodoulou et al. 2010], significantly lower than the best known upper bound $(n+1)/2$. The bottleneck of applying the techniques of the current work to these variants appears to be the lack of a good characterization of 2×2 fractional mechanisms. Another important direction is to apply our approach to major open questions in other settings and in particular to combinatorial auctions.

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