Randomized Apportionment Methods for Exact Proportional Representation: a Short Survey

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Apportionment is the problem of allocating seats to political parties in a parliament in proportion to their deserved representation or to allocate the number of representatives to states in proportion to their size. Throughout history, most of the focus is on deterministic methods to apportion seats among the groups which may favour bigger or smaller parties or may have some inherent mathematical limitations. We survey various randomized rules that achieve exact apportionment in expectation.

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1. INTRODUCTION

Suppose there are three towns of sizes 30k, 80k and 90k from where a total of two representatives will be selected. Should the town with the largest population get two representatives? Or should the two most populous towns get one representative each? Or should the least populous town also have some chance of getting a representative? These types of questions concern what is called *apportionment*.

In apportionment, n disjoint groups are to be allocated a given number of slots k in proportion to the group sizes. The problem is ubiquitous in settings such as proportional representation of seats in the US congress, European Parliament, and the German Bundestag as well as various other committee selection settings [Balinski and Young, 1980, Birkhoff, 1976, Mayberry, 1978, Niemeyer and Niemeyer, 2008]. It has been studied in political science, economics, operations research and computer science. Apportionment is one of the most well-studied problems in social choice and political science with various books written on the topic [Balinski and Young, 1982, Pukelsheim, 2014, Young, 1994]. The classic book of Balinski and Young [1982] discusses history of apportionment rules used in the American Congress.

> *"This surprisingly difficult problem has concerned statesmen, political analysts and mathematicians for over two hundred years. The reason is the central importance of that apportionments plays in representative government. The difference of just one seat can be crucial in tipping the balance of power in a legislature. Hence the design of apportionment formulas is of abiding interest to politicians."*

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Balinski and Young add that the problem is applicable to many other scenarios:

> *"Similar problems arise in many other settings, however. Teachers are assigned to courses in proportion to the number of students who register for them. Medical personnel are assigned to army units in proportion to the number of soldiers in each unit. Computers and support staff are allocated to divisions in a firm according to the measures of need and demand."*

In the settings mentioned above, each party (also called abstractly as group) i has relative size x_i with $\sum_{i=1}^{n} x_i = 1$. In many settings, the input is not the relative sizes but the actual group populations or group entitlements. Since the goal is proportional representation, each group requires kx_i slots which is termed as its *target quota* or *entitlement* q_i for group i:

$$q_i = kx_i.$$

Since $q_i = kx_i$ may not be an integer, we have to resort to *apportionment* which means that in order to allocate exactly k slots, some group may get slightly more or less that its target quota. We call the outcome $t = (t_1, \dots, t_n)$ where $\sum_{i \in [n]} t_i = k$ where each t_i is an integer. A minimal requirement is that the outcome should satisfy *quota compliance*: each group should get quota that is the result of the target quota being rounded up or down: $t_i \in \{\lfloor q_i \rfloor, \lceil q_i \rceil\}$ for all $i \in [n]$.

Example 1 Apportionment Problem. *Suppose $q = (q_1, q_2, q_3) = (0.3, 0.8, 0.9)$. We wish to select $k = 2$ seats. One outcome is $t = (t_1, t_2, t_3) = (0, 1, 1)$ in which the last two entries are rounded up and the first entry is rounded down. An ex-post integral outcome is one in which two entries are rounded up and one entry is rounded down.*

Several apportionment procedures have been introduced in the literature such as the methods of Hamilton, Jefferson, Webster, Adams, and Hill [Balinski and Young, 1982]. Hamilton's method is essentially the method of the 'largest remainders' whereby each party is given its lower quota and then parties with the largest remainders are then given an extra seats to achieve the target seats. Many of the other methods such as Jefferson, Webster, Adams, and Hill are *divisor methods* in which each quota is divided by a given divisor such that when a rounding function is applied to each of the terms and then all the rounded terms are added, we get the desired number of seats. Different rounding functions correspond to different divisor methods. Most of the apportionment method discussed in theory and applied in practice are deterministic methods and each of them has some drawbacks. For example some method favours the bigger groups whereas some other favours the smaller groups. One particular disadvantage is not a design flaw of the method but is a consequence of a fundamental mathematical impossibility concerning deterministic methods. Balinski and Young [1982] proved that *no* deterministic apportionment procedure can simultaneously satisfy the following two of axioms: (1) *quota compliance* and (2) *population monotonicity* (if the ratio between the entitlements of two states i and j increases then it should be the case that the number of seats of i increases and the the number of seats of j decreases). However, if we use randomization, we can achieve the target quotas exactly (in expectation) which means we can satisfy the two axioms. Randomization also has another benefit. It can ensure that every group has at least some probability of being represented. For example, in the three town example, we can ensure that each town has some probability of having a selected representative.

Note that any randomized method that achieves in expectation the expected target quota for each group also satisfies the following properties in expectation: (1) *Bias Condition*: The apportionment method should be free of bias, that is, it should neither favour large nor small parties' seats (2) *Independence Condition* (the number of seats assigned to the party depends only on its exact quota but not on the distribution of the quotas of other parties); (3) *House monotonicity* (if k increases then the seats of no party decreases).

How can we use randomization to achieve exact proportional representation in expectation and also achieve quota compliance ex post? We take a tour and visit some of the randomized apportionment methods that have been proposed in the literature.

2. A TOUR OF SOME RANDOMIZED APPORTIONMENT METHODS

We discuss some of the randomized apportionment methods.

2.1 Grimmet's Method

Grimmet [2004] presented a randomized apportionment rule that achieves the quota requirements. Grimmet's algorithm requires two successive randomized decisions, and one of them involves a continuous variable. The method satisfies the quota rule as do all the other randomized rules that we will discuss.

Grimmet's Method

(1) Chooses a permutation of the groups uniformly at random.
(2) Draw U uniformly at random from $[0,1]$, and let $Q_i = U + \sum_{j=1}^{i} q_j$. Allocate to each group i one seat for each integer contained in the interval $[Q_{i-1}, Q_i)$.

Note that $\sum_{j=1}^{i} q_i = k$. So the number of integers between U and $\sum_{j=1}^{i} q_i + U$ is k as well. Next, we observe that a particular group j's allocation is the number of integers between $U + \sum_{j=1}^{i-1} q_j$ and $U + \sum_{j=1}^{i-1} q_j + q_i$. This is either $\lfloor q_i \rfloor$ or $\lceil q_i \rceil$ so quota compliance is satisfied.

Example 2 Grimmet's Method. $q = (0.3, 0.8, 0.9)$ *We wish to select* $k = 2$ *additional seats. We choose a permutation uniformly at random and suppose it is* 123. *We draw* U *uniformly at random from* $[0,1]$ *and suppose it is* 0.2. *Then* $Q_0 = 0.2$, $Q_1 = 0.5$, $Q_2 = 1.3$, $Q_3 = 2.2$. *Then, the ex post outcome is* $r = (0,1,1)$ *that gives one seat each to the second and third party. The outcome can be different depending on what is the drawn value of* U.

2.2 Pipage Method

We describe the randomized method called Pipage that has proposed in mathematics and computer science [Gandhi, Khuller, Parthasarathy, and Srinivasan, 2006, Deville and Tille, 1998, Ageev and Sviridenko, 2004]. The method can be viewed as starting from the root node of a tree and iteratively taking one of the two possible branches with particular probabilities. As we traverse down the tree, the number of fractional entries decreases by one in each step. In at most n steps, we reach a fully integral rounded vector. The probability with which various integral vectors are generated gives in expectation the original vector $(q_i)_{i\in[n]}$.

Pipage Method

Let $q^0 = q$. We iteratively and randomly modify q^0 in rounds. Denote $q^t = (q_1^t, q_2^t, ..., q_m^t)$ as the values at round t. In each round, we update the values of at most two indices while keeping the values of all other indices constant. Let $F^t = \{i \in C \mid q_i^t \in (0,1)\}$ be the set of indices that are fractional in round t. The update rule depends on the cardinality of F^t.
While $|F^t| \geq 2$, we arbitrarily select two indices $i, j \in F^t$ and run the following randomized update rule:

$$(q_i^{t+1}, q_j^{t+1}) = \begin{cases} \left(q_i^t + a, q_j^t - a\right) & \text{with probability} \quad \frac{b}{a+b} \\ \left(q_i^t - b, q_j^t + b\right) & \text{with probability} \quad \frac{a}{a+b} \end{cases}$$

where

$$a = \min\{c > 0 \mid q_i^t + c = 1 \text{ or } q_j^t - c = 0\}$$

and

$$b = \min\{c > 0 \mid q_i^t - c = 0 \text{ or } q_j^t + c = 1\}.$$

For all other indices $\ell \in C \setminus \{i, j\}$, we set $q_\ell^{t+1} = q_\ell^t$.
If $|F^t| < 2$, we terminate the algorithm and set $P_i = q_i^t$ for all $i \in C$.

Let us explain a few salient features of the method. At every step, when two entries are updated, the total amount in the vector remains the same: if one entry is decreased, the other entry is increased by the same amount. Now if one entry x is increased by a and decreased by b in the other branch, then it is increased by a with probability $b/(a+b)$ and decreased by b with probability $a/(a+b)$. Hence the expected net change is $ab/(a+b) - ab/(a+b) = 0$ as desired.

Example 3 Pipage Method. *Let us illustrate the Pipage method.* $q = (0.3, 0.8, 0.9)$. *We wish to select* $k = 2$ *seats. Then, the method works as follows in Figure 2.2. The probability of each outcome is equal to the product of probabilities along the path from the root node. For example, the probability of outcome* $(1,1,0)$ *is*
$\frac{7}{9} \times \frac{1}{10} + \frac{2}{9} \times \frac{1}{10} = \frac{9}{90} = \frac{1}{10}$.

It is clear the Pipage method does not construct an explicit probability distribution over a polynomial number of integral allocations. In each step, it branches in two directions and possibly has an exponential number of integral allocations in its support. However a single polynomial-sized path of the decision tree is traversed to generate an integral allocation.

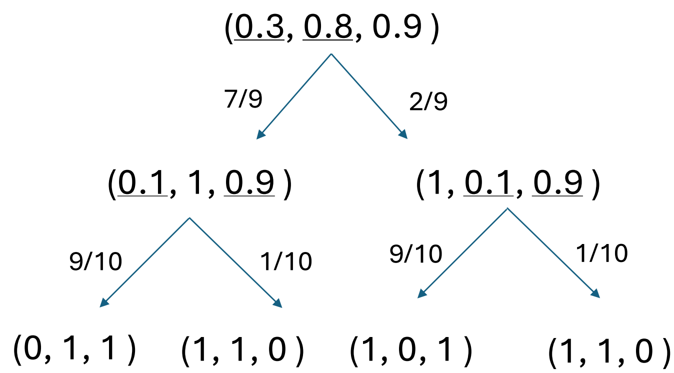

Fig. 1. Pipage Method. At each node except the leaf nodes, the underlined indices are selected and then a randomized update rule is run on them.

2.3 Careful Sliding Method

Next, we present the Careful Sliding Method. It has been presented as a subroutine for a 'peer selection' mechanism [Aziz, Lev, Mattei, Rosenchein, and Walsh, 2019] but it constitutes a simple and easy to apply rule for randomized apportionment. One feature of the method is that it returns an explicit probability distribution over integral apportionment and the size of the distribution is linear in the input size. The algorithm starts by rounding up the lowest quotas to obtain an integral allocation. The probability assigned to such an allocation is maximized subject to the condition that there is enough probability with which the higher quotas can also be rounded up to achieved their target quota in expectation. Once a given group has already achieved its expected quota, it is made *tight* and only the next higher groups are considered for rounding up. In the other direction, if there is a group with a high target quota and it needs to be rounded up in all subsequent integral allocations, it is made tight as well. In each step, a low quota or a high quota group is made tight. A low quota that is made tight is *not* rounded up in any subsequently generated integral allocations. A high quota that is made tight is rounded up in *all* subsequently generated integral allocations.

Before formally describing the method, we first present a simple illustrative example for how it works.

Example 4. *Suppose* $q = (0.3, 0.8, 0.9)$. *We wish to select* $k = 2$ *seats. The outcome of the Careful Sliding Method is as follows. The integral allocations and their corresponding probabilities are listed.*

$$S_1 = (1, 1, 0) : 0.1$$
$$S_2 = (1, 0, 1) : 0.2$$
$$S_3 = (0, 1, 1) : 0.7$$

The lottery computed indicates that the probability of outcome $(1, 1, 0)$ *is* 0.1*; the probability of outcome* $(1, 0, 1)$ *is* 0.2*; and the probability of outcome* $(0, 1, 1)$ *is* 0.7*.*

In the first step, we note that since $0.9 < 1$*, it does not need to be rounded up in the current and all subsequently generated integral outcomes. Hence, it is not labeled tight and will not be rounded up in the currently generated integral outcome. So we round the first two non-tight numbers* 0.3 *and* 0.8 *to get* $S_1 = (1, 1, 0)$*. The maximum probability that can be assigned to* $(1, 1, 0)$ *is* $1 - 0.9 = 0.1$ *Any higher probability makes it impossible to achieve* 0.9 *in expectation for the last entry. After* S_1 *and its corresponding probability are generated, we know that* 0.9 *will always be rounded up in* $S_2, S_3, \ldots$*, so it is labeled tight. So* $S_2 = (1, 0, 1)$ *is assigned probability* 0.2 *as giving any more probability does not leave enough probability for the second party to achieve its expected quota of* 0.8*. After this,* 0.3 *is labeled tight and the only entry that is not tight is* 0.8*. Finally,* $S_3 = (0, 1, 1)$ *with corresponding probability* 0.7*.*

Next, we present the method's description.

Careful Sliding Method

(1) Allocate to each group i, its lower quota $\lfloor q_i \rfloor$ to reduce the problem to that of allocating $\alpha = k - \sum_{i \in N} \lfloor q_i \rfloor$ seats with target $s_i = q_i - \lfloor q_i \rfloor$ for each group.

(2) **Initialize:** Label s_i for all i as *non-tight*. Set unallocated probability r to be 1. Set $\ell = 0$.
(we want to gradually relabel all s_i's as tight. We will iteratively generate a new integral outcome $S_{\ell+1}$ and its corresponding probability $p_{\ell+1}$ to achieve a partial lottery $[(S_1 : p_1), (S_2 : p_2) \ldots, (S_{\ell+1} : p_{\ell+1})]$.)

(3) Check if for highest non-tight number (say s_j), it is the case that the expected value of s_j can be obtained if we round up s_j in all $S_{\ell+1}, \ldots$. If yes, we decrement α by one and set s_j to be *tight*. The entry s_j will be rounded up in allocations $S_{\ell+1}, \ldots$. Repeat until $\alpha = 0$ or the condition does not hold.

(4) Increment ℓ by one. Round up the smallest non-tight α numbers to get S_ℓ. The corresponding probability p_ℓ of S_ℓ is the maximum feasible probability such that it still allows enough unallocated probability $r - p_\ell$ to achieve the highest non-tight number. Add p_ℓ and the corresponding rounded outcome S_ℓ to the distribution. Set r to $r - p_\ell$. If some s_i is already achieved in partial lottery $[(S_1 : p_1), \ldots, (S_\ell : p_\ell)]$, then it cannot be rounded up in future allocations so, its relabelled as tight (will be rounded down in all next allocations $S_{\ell+1}, \ldots$).

(5) Repeat (3), (4) until $\alpha = 0$.

(6) Return distribution $[(S_1 : p_1), \ldots, (S_\ell : p_\ell)]$.

2.4 Sampford's Method

Finally, we describe a simple method due to Sampford [1967] that has recently been shown to satisfy desirable axiomatic properties. The method first allocates each group i, its lower quota $\lfloor q_i \rfloor$ to reduce the problem to that of allocating $\alpha = k - \sum_{i \in N} \lfloor q_i \rfloor$ seats with target $p_i = q_i - \lfloor q_i \rfloor$ for each group. The first party to be given an extra seat is selected with probability proportional to p_i. For all the subsequent draws, parties are drawn with probabilities proportional to $\frac{p_i}{1-p_i}$. If any of the α groups is repeated, we restart to sample α groups.

Sampford's Method

(1) Already allocate each group i, its lower quota $\lfloor q_i \rfloor$ to reduce the problem to that of allocating $\alpha = k - \sum_{i \in N} \lfloor q_i \rfloor$ seats with target $p_i = q_i - \lfloor q_i \rfloor$ for each group.
(2) Select α groups with replacement, the first drawing being made with probabilities p_i for each group i and all the subsequent ones with probabilities proportional to $\frac{p_i}{1-p_i}$.
(3) **If** the α drawn parties are distinct, give one additional seat to each of them;
Else, start over.

One important issue is that the method may not terminate as we could repeatedly get a party that gets two of the α seat which violates quota compliance. However, the sampling can also be implemented in a polynomial-time manner [Grafström, 2009] although the polynomial-time algorithm's description is not as simple as the original method's. An alternative and simpler method than Sampford is *weighted random sampling* in which each of the α parties is selected with probability proportional to p_i. However such a naive way to sample does not respect that each party's seats are rounded up with probability p_i.

EXAMPLE 5 SAMPFORD'S METHOD. $p = (0.3, 0.8, 0.9)$ *We wish to select* $k = 2$ *additional seats. We first select the first party with probabilities proportional to* $0.3, 0.8, 0.9$ *respectively. Suppose the first party selected is the third one. One of the parties from party 1, party 2, and party 3 are selected with probabilities proportional to* $0.3/0.7$, $0.8/0.2$, *and* $0.9/0.1$ *respectively. Equivalently, party 1 is selected with probability* $(3/7)/((3/7)+4+9)$, *party 2 is selected with probability* $4/((3/7)+4+9)$, *and party* 3 *is selected with probability* $9/((3/7) + 4 + 9)$.

3. DISCUSSION AND RESEARCH DIRECTIONS

We have illustrated key randomized apportionment methods. The growing literature on randomized apportionment fits in the wider framework of "*best of both worlds fairness*" whereby the goal is to simultaneously achieve strong fairness properties in expectation and approximately fair properties ex-post [Aziz, Freeman, Shah, and Vaish, 2024].

Recently, there has been a surge in designing or identifying apportionment rules that satisfy further axiomatic properties [Gölz, Peters, and Procaccia, 2022, Correa, Gölz, Schmidt-Kraepelin, Tucker-Foltz, and Verdugo, 2024]. For example, Gölz et al. [2022] use the techniques of Gandhi et al. [2006] to additionally achieve a strong specification of *house monotonicity* (if the number of seats increases, no one's seats are reduced).

Although randomized apportionment achieves population monotonicity in expectation, when parties or voters care about joint events, such as whether a coalition of parties reaches a majority, further care needs to be made when generating a random outcome. Correa et al. [2024] observe that when parties already get their lower quotas, the decision about apportioning the remainders reduces to *"probability proportional to size" sampling without replacement* that is well-studied in mathematical statistics [Brewer and Hanif, 2013] and has dozens of well-established methods. They focus on *threshold monotonicity* that requires that for any coalition of parties T, if the tarqet quota of each party in T weakly increases and each party outside T weakly decreases, then the random variable describing the total number of seats awarded to T first-order stochastically dominates the corresponding random variable before the reinforcement. Correa et al. [2024] showed that the Sampford's Method is especially suitable as it satisfies a weaker version of threshold monotonicity. On the other hand, the other methods in this exposition violate selection monotonicity. Correa et al. [2024] ask whether Sampford's Method or some other method satisfies threshold monotonicity.

The recent results indicate that although apportionment has been studied for centuries, there are still interesting ongoing developments to understand randomized apportionment methods and their axiomatic properties. Future work on randomized apportionment promise to have a similar trajectory as the well-established literature on deterministic apportionment rules [Balinski and Young, 1982, Pukelsheim, 2014, Young, 1994]. This will entail several important steps including

- expanding the toolkit of apportionment rules and classifying them into various families;
- formalizing axiomatic properties for rules;
- understanding which rules satisfy what properties; and
- understanding what subsets of axioms are compatible or impossible to satisfy simultaneously.

For the multitude of axioms considered in the literature on deterministic rules, there may be multiple ways to generalize them to the rich class of randomized rules. Finally, axiomatically characterizing desirable randomized apportionment rules with respect to various axiomatic properties remains a major open question.

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REFERENCES

A. A. Ageev and M. I. Sviridenko. Pipage rounding: A new method of constructing algorithms with proven performance guarantee. *Journal of Combinatorial Optimization*, 8(3):307–328, 2004. doi: 10/dv46g4.

H. Aziz, O. Lev, N. Mattei, J. S. Rosenchein, and T. Walsh. Strategyproof peer selection using randomization, partitioning, and apportionment. *Artificial Intelligence*, 275:295–309, 2019.

H. Aziz, R. Freeman, N. Shah, and R. Vaish. Best of both worlds: Ex ante and ex post fairness in resource allocation. *Operations Research*, pages 1317–1750, 2024.

M. Balinski and H. P. Young. *Fair Representation: Meeting the Ideal of One Man, One Vote*. Yale University Press, 1982. (2nd Edition [with identical pagination], Brookings Institution Press, 2001).

M. L. Balinski and H. P. Young. The webster method of apportionment. *Proceedings of the National Academy of Sciences (PNAS)*, 77(1):1–4, 1980.

G. Birkhoff. House monotone apportionment schemes. *Proceedings of the National Academy of Sciences (PNAS)*, 73(3):684–686, 1976.

K. RW Brewer and M. Hanif. *Sampling with unequal probabilities*, volume 15. Springer Science & Business Media, 2013.

J. Correa, P. Gölz, U. Schmidt-Kraepelin, J. Tucker-Foltz, and V. Verdugo. Monotone randomized apportionment. *CoRR*, abs/2405.03687, 2024.

J-C. Deville and Y Tille. Unequal probability sampling without replacement through a splitting method. *Biometrika*, 85(1):89–101, 1998. doi: 10.1093/biomet/85.1.89.

R. Gandhi, S. Khuller, S. Parthasarathy, and A. Srinivasan. Dependent rounding and its applications to approximation algorithms. *Journal of the ACM*, 53(3):324–360, 2006.

P. Gölz, D. Peters, and A. D. Procaccia. In this apportionment lottery, the house always wins. In *The 23rd ACM Conference on Economics and Computation (ACM EC)*, page 562. ACM, 2022.

A. Grafström. Non-rejective implementations of the Sampford sampling design. *Journal of Statistical Planning and Inference*, 139(6):2111–2114, 2009.

G. Grimmet. Stochastic apportionment. *The American Mathematical Monthly*, 111 (4):299–307, 2004.

J. P. Mayberry. Quota methods for congressional apportionment are still non-unique. *Proceedings of the National Academy of Sciences (PNAS)*, 75(8):3537–3539, 1978.

H. F. Niemeyer and A. C. Niemeyer. Apportionment methods. *Mathematical Social Sciences*, 56:240–253, 2008.

F. Pukelsheim. *Proportional Representation: Apportionment Methods and Their Applications*. Springer, 2014.

Michael R. Sampford. On sampling without replacement with unequal probabilities of selection. *Biometrika*, 54(3-4):499–513, 1967. doi: 10/ccb3kz.

H. P. Young. *Equity: in Theory and Practice*. Princeton University Press, 1994.