Duality in Combinatorial Auctions

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A combinatorial auction is a way of allocating multiple indivisible units of heterogeneous objects from a single seller to two or more buyers. Efficient combinatorial auctions can be described in terms of the duality theory of linear programming. In fact, there may exist several LP formulations of the same problem that differ with respect to whether or not the prices in the dual vary with the buyer (non-anonymous) and/or the collection of objects purchased (non-linear).

Because of its well-known incentive properties, we focus on the Vickrey auction. These are efficient auctions in which buyers pay the social opportunity cost of their purchases and consequently are rewarded with their marginal product (i.e., the increase in the gains from trade from their participation in the auction).

The efficient allocation of objects may be obtained as a solution to an LP problem without any qualifications on buyers' utilities. The dual problem yields the set of *pricing equilibria*. At any pricing equilibrium in the underlying exchange economy, the seller (auctioneer) and the buyers maximize with respect to prices and markets clear. These prices may be nonlinear and non-anonymous: this is the key difference between a pricing equilibrium and the linear anonymous pricing of Walrasian equilibrium. It is important to observe that Vickrey payments are typically nonlinear and non-anonymous even when the auction permits an LP formulation with Walrasian pricing.

There exists a pricing equilibrium at which buyers obtain their marginal products — and, therefore, are rewarded according to a Vickrey auction — if and only if *buyers are substitutes* (see Bikhchandani and Ostroy [2002]). The buyers are substitutes condition states that the marginal product of buyers is superadditive: the marginal product of a subset of buyers is at least as large as the sum of the marginal products of the buyers in that subset.

Buyers are substitutes is a synthetic condition involving all buyers. It holds if each buyer's utility function satisfies a generalization of Kelso and Crawford [1982]'s gross substitutes condition. Murota [2003] has demonstrated that the generalized gross substitutes condition is a characterization of discrete concavity. The dual characterization of a discretely concave function, to be noted below, is that its subdifferential has the lattice property. Because gross substitutes is almost necessary for the buyers are substitutes condition, the conclusion that Vickrey payments can be implemented as a pricing equilibrium practically requires that each buyer's utility is discretely concave.

Algorithms for solving LP problems imply that the static or sealed-bid characterization has a "dynamic" or an ascending-price counterpart. The primal-dual algorithm is important because its informational requirements can be decentralized. Without knowledge of buyer preferences, the implementor can act as an auctioneer who calls out prices — otherwise known as feasible solutions to the dual. Using only knowledge of one's own preferences, each buyer makes utility maximizing bids based on those prices; and the auctioneer/implementor uses the information from those bids to adjust prices.

Ausubel [2004] formulated an ascending-price auction for homogeneous goods to implement a Vickrey scheme, which can be described as nonlinear version of the Law of Supply and Demand. Bikhchandani and Ostroy [2006] show that the indivisible commodity version of Ausubel's auction is a primal-dual algorithm on the above mentioned LP formulation. de Vries et al. [2007] construct an incentive-compatible primal-dual auction for heterogenous objects which builds upon ideas in Demange et al. [1986] and the LP formulation in Bikhchandani and Ostroy [2002].

It is noteworthy that a Walrasian equilibrium (i.e., a pricing equilibrium with linear, anonymous prices) exists in a homogeneous goods model with diminishing marginal utility (Ausubel's setting) and in a heterogeneous goods model with gross substitutes (de Vries et al.'s setting). In fact, because buyer utility functions are in both cases discretely concave, Walrasian prices have the lattice property, and therefore a smallest Walrasian price vector, the one most favorable to the buyers, exists. Nevertheless, the payments made by buyers at a smallest Walrasian price vector are typically too large to implement the Vickrey auction.

The results, above, graft two related ideas to form a useful hybrid. In linear programming, the primal-dual algorithm was originally developed for the assignment model. In economics, the algorithm is the Law of Supply and Demand, i.e., the story of how market prices adjust to establish equilibrium. In a combinatorial ascending-price auction, the metaphorical Auctioneer of the market is replaced by a literal entity, as well as a detailed procedure adjusting prices to bids. A complicating factor in the auction problem, ignored in the Law of Supply and Demand, is that bidders should have an incentive to bid truthfully. These incentive requirements conflict with the linearity and anonymity of pricing with which the Law is normally identified. Nevertheless, the logic of prices adjusting to excess demands can be extended (provided buyers are substitutes) beyond its traditional boundaries to meet the added challenge of incentive-compatible pricing.

If the buyers are substitutes condition is not satisfied then, as noted above, a marginal product pricing equilibrium does not exist. Therefore, an ascending-price version of the Vickrey auction cannot be implemented via a pricing equilibrium. Parkes and Ungar [2000], Ausubel and Milgrom [2002], and Mishra and Parkes [2007] have proposed ascending-price auctions that converge to an efficient allocation whether or not buyers are substitutes. Interestingly, these auctions satisfy ex post incentive compatibility only when buyers are substitutes. The existence of efficient, incentive-compatible, ascending-price auctions when buyers are not substitutes is an open question.

REFERENCES

AUSUBEL, L. M. 2004. An efficient ascending-bid auction for multiple objects. *American Economic Review 94*, 1452–1475.

AUSUBEL, L. M. AND MILGROM, P. 2002. Ascending auctions with package bidding. *Frontiers of Theoretical Economics 1*, 1–42.

BIKHCHANDANI, S. AND OSTROY, J. M. 2002. The package assignment model. *Journal of Economic Theory 107*, 377–406.

BIKHCHANDANI, S. AND OSTROY, J. M. 2006. Ascending price vickrey auctions. *Games and Economic Behavior 55*, 215–241.

DE VRIES, S., SCHUMMER, J., AND VOHRA, R. 2007. On ascending vickrey auctions for heterogeneous objects. *Journal of Economic Theory 132*, 95–118.

DEMANGE, G., GALE, D., AND SOTOMAYOR, M. 1986. Multi-item auctions. *Journal of Political Economy 94*, 863–872.

KELSO, A. AND CRAWFORD, V. 1982. Job matching, coalition formation, and gross substitutes. *Econometrica 50*, 1483–1504.

MISHRA, D. AND PARKES, D. C. 2007. Ascending price vickrey auctions for general valuations. *Journal of Economic Theory 132*, 335–366.

MUROTA, K. 2003. *Discrete Convex Analysis*. SIAM, Philadelphia, PA.

PARKES, D. C. AND UNGAR, L. H. 2000. Iterative combinatorial auctions: Theory and practice. In *Proc. 17th National Conference on AI*. 74–81.