# Better Mechanisms for Combinatorial Auctions via Maximal-in-Range Algorithms?

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Algorithmic Mechanism Design attempts to design algorithms that handle the strategic behavior of selfish players. Many of the problems considered in the field involve allocation of resources to players, and the paradigmatic abstraction is that of combinatorial auctions. In a combinatorial auction we have $n$ bidders and $m$ items. Each bidder $i$ has a valuation function $v_i$ that gives some non-negative value to each possible subset of the items. We assume that the valuations are monotone, and that for each $v_i$ we have that $v_i(\emptyset) = 0$. In this note our goal is to find a partition of the items $S_1, ..., S_n$ such that the total social welfare, $\Sigma_i v_i(S_i)$, is maximized. Similarly to most recent work in algorithmic mechanism design, our goal is to design *truthful* mechanisms. I.e., mechanisms where the dominant strategy of each bidder is to reveal his true valuation. We require our algorithms to run in time that is polynomial in $n$ and $m$, the natural parameters of the problem.

From a purely computational point of view, there exists an $O(\sqrt{m})$-approximation algorithm for this problem. This is the best ratio that can be obtained in polynomial time. See [Blumrosen and Nisan 2007] for a recent survey. From a game theoretic point of view, the classic VCG mechanism can be used to allocate the items in a truthful manner. However, the VCG mechanism involves finding an optimal solution – a computationally unfeasible task. One of the most important questions in algorithmic mechanism design is to determine the best approximation ratio for this problem that is achievable by polynomial time truthful mechanisms[1].

We suggest tackling this question by using VCG-based mechanisms. Recall that in the VCG payment scheme we obtain an optimal solution $(O_1, \dots, O_n)$, and allocate accordingly. Each bidder $i$ is being paid $\Sigma_{j \neq i} v_i(O_i)$. Thus, the utility of each bidder $i$ equals to the value of the optimal solution: $v_i(O_i) + \Sigma_{j \neq i} v_i(O_i)$. It is not hard to see that truthfulness is a dominant strategy for each bidder. What about the running time? At a first glance, it looks that VCG does not help much in constructing polynomial-time mechanisms, as we have already mentioned that VCG requires finding the optimal solution, and that finding the optimal solution

[1]We restrict our discussion to deterministic mechanisms. If randomization is allowed then much more is known [Dobzinski 2007; Dobzinski et al. 2006; Lavi and Swamy 2005]

is computationally hard. One naive solution might be to use an approximation algorithm that finds an approximate solution $(S_1, \ldots, S_n)$, and uses VCG payments with respect to this approximate solution. I.e., the mechanism pays each bidder $i$ $\Sigma_{j \neq i} v_i(S_i)$. Unfortunately, in general using VCG payments together with an arbitrary approximation algorithm does not result in a truthful mechanism. There are, however, some approximation algorithms that are truthful when the VCG payment scheme is used:

**Definition:** An algorithm is called *maximal in range* (MIR) if there exists a subset $\mathcal{R}$ of allocations (the range of the algorithm), such that for every possible input $v_1, \ldots, v_n$, the algorithm outputs the allocation that maximizes the welfare in $\mathcal{R}$. I.e., for all input valuations $v_1, \ldots, v_n$ the algorithm outputs $\arg\max_{(S_1,\ldots,S_n)\in\mathcal{R}} \Sigma_i v_i(S_i)$.

MIR algorithms were studied in [Nisan and Ronen 2000]. Informally, Nisan and Ronen prove that MIR algorithms are the only algorithms that become truthful using the VCG payment scheme. MIR algorithms might look very simple, but in some settings they are quite powerful. Perhaps the most notable example is the case of multi-unit auctions, where a 2-approximation MIR algorithm exists. Furthermore, in some settings the latter algorithm can be altered to obtain a PTAS [Dobzinski and Nisan 2007b]. Another example is the case of combinatorial auctions with subadditive bidders, where a ratio of $O(\sqrt{m})$ can be achieved using MIR algorithms [Dobzinski et al. 2005]. Surprisingly, MIR algorithms were also used to design approximation algorithms, not necessarily truthful; for example, Arora's PTAS for TSP in the plane [Arora 1998] is in fact maximal in its range. For combinatorial auctions with general bidders, the subject of this letter, the best currently known deterministic truthful mechanism obtains a ratio of $O(\frac{m}{\sqrt{\log m}})$ and is maximal in its range [Holzman et al. 2004].

**Main Open Question:** Can a polynomial-time maximal-in-range algorithm provide an approximation ratio of $O(\sqrt{m})$ for combinatorial auctions with general bidders?

Of course, a positive answer will immediately provide us with an optimal truthful mechanism for combinatorial auctions. To get some intuition regarding the open question and MIR algorithms in general, let us consider the mechanism of [Holzman et al. 2004]. This mechanism partitions the items into $\log m$ arbitrary bundles, each of size $\frac{m}{\log m}$, and allocates the items according to the best allocation of the whole bundles to the bidders. This algorithm is clearly maximal in range. The algorithm can be shown to run in polynomial time by recalling that an optimal allocation can be found by dynamic programming in time that is polynomial in the number of bidders and exponential in the number of items. In our case, since only whole bundles are allocated, the running time is polynomial in $n$, and in $2^{\log m} = m$, as needed. To give some intuition regarding the claimed approximation ratio, consider an instance with $\frac{m}{\sqrt{\log m}}$ bidders. In this instance the items can be partitioned to bundles $S_1, \ldots, S_n$, each of size $\sqrt{\log m}$, such that $v_i(S) = 1$ if $S$ contains $S_i$, and $v_i(S) = 0$ otherwise. Observe that in order to satisfy one bidder, the algorithm might have to allocate him $\frac{m}{\sqrt{\log m}}$ items, by allocating $\sqrt{\log m}$ bundles of size $\frac{m}{\log m}$.

Thus, by satisfying one bidder we might not be able to satisfy $\frac{m}{\sqrt{\log m}}$ other bidders, hence the approximation ratio. The reader is encouraged to convince himself that this is essentially the worst-case example for this algorithm.

How can we improve upon this simple algorithm? Perhaps the first idea that comes into mind is to partition the items in a smarter way. More generally, consider the following class of algorithms: each bidder $i$ provides his value for each bundle $S \subseteq \mathcal{A}_i$, where $\mathcal{A}_i$ is a set of bundles that is pre-determined independently of the input. The algorithm finds the best feasible solution $(S_1, \ldots S_n)$, where each $S_i \in \mathcal{A}_i$. In other words, the valuation of each bidder is "flattened", and the best allocation is chosen, with respect to the flattened valuations. While all maximal-in-range algorithms in the literature work this way, including the algorithm of [Holzman et al. 2004], this class will not take us too far in our setting: algorithms in the class can be implemented using value queries only[2], but value queries cannot provide an approximation ratio better than $\Omega(\frac{m}{\log m})$ for combinatorial auctions with general bidders in polynomial time [Blumrosen and Nisan 2007]. It seems plausible that a better algorithm would require us to develop new techniques for constructing MIR algorithms, possibly by considering iterative MIR algorithms.

One way of escaping this impossibility result is by restricting the bidders to be $k$-minded: bidder $i$'s valuations is defined by (at most) $k$ pairs $(S_1, t_1), \ldots (S_k, t_k)$. Let $v_i(S) = \max_r\{t_r | S_r \subseteq S\}$. To escape the $\Omega(\frac{m}{\log m})$ lower bound we assume that in each pair $(S_r, t_r)$, the bundle $S_r$ is publicly known (and the value $t_r$ is private information). What is the power of MIR algorithms in this setting? Even if all bidders are single-minded ($k = 1$), the answer is unknown. Furthermore, the currently best truthful mechanism (not just MIR) for this setting, even for $k = 2$, is the $O(\frac{m}{\sqrt{\log m}})$-approximation mechanism of [Holzman et al. 2004]! As an easy warm-up, the reader is encouraged to prove that given an $MIR$ $\alpha$-approximation algorithm for single-minded bidders, we can construct a $(k \cdot \alpha)$-approximation algorithm for $k$-minded bidders.

Another possible direction is to prove that MIR algorithms do not have much power in the setting of combinatorial auctions with general bidders. For combinatorial auctions with subadditive bidders, a lower bound of $m^{\frac{1}{6}}$ for approximating the welfare using MIR algorithms exists [Dobzinski and Nisan 2007a]. This result is meaningless for general bidders, not necessarily subadditive – even from a purely computational perspective, without considering incentives at all, a lower bound of $\Omega(m^{\frac{1}{2-\epsilon}})$ is known. However, it might be possible to improve and extend the results of [Dobzinski and Nisan 2007a] to the case of general bidders.

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[2]A value query specifies a bundle $S$ of items and receives $v(S)$ as a reply.

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