Characterizing Truthfulness In Discrete Domains

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Algorithmic mechanism design [9; 10] focuses on the design of algorithms that aim to achieve global objectives in settings in which the "input" is provided by self-interested strategic players[2]. This necessitates the design of algorithms that are *incentive-compatible* (a.k.a. *truthful*[3]) in the sense that players are incentivized via payments to behave as instructed. The most natural approach to designing incentive-compatible algorithms is coming up with an algorithm *and* an explicit payment scheme that guarantees its incentive-compatibility. However, finding appropriate payments is often a difficult, setting-specific, task, which is mostly achievable for very simple types of algorithms.

A more general approach is the following: Any algorithm that interacts with selfish players and then outputs an outcome, can be regarded as computing a function, called a *social-choice function*, from the players' "input" to some outcome space. Certain properties of social-choice functions are known to imply their *implementability*, that is, the *existence* of a payment scheme that guarantees incentive-compatibility. Hence, instead of explicitly dealing with payments, the problem of designing incentive-compatible algorithms boils down to analyzing the mathematical properties of the social-choice functions computed by algorithms. This approach makes sense if these mathematical properties are simple and easy to analyze.

A simple constraint on social choice functions called "*weak-monotonicity*" has been shown to characterize the implementability of social choice functions in several interesting settings. However, with the exception of very restricted settings named "single parameter domains" [1], all these characterizations of incentive-

[1]Supported by grants from the Israel Science Foundation.
[2]We deal with the standard quasilinear mechanism design setting.
[3]Formally, the solution concept we consider is incentive-compatibility in ex-post Nash (in particular, our results hold for incentive-compatibility in dominant strategies, which is a special case of this solution concept).

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compatibility are known to apply only to environments in which the private information of the players is drawn from inherently non-discrete domains, like convex domains (see [2; 4; 12; 8]). Our work [13] is motivated by the fact that in many cases the private information of the players is drawn from discrete domains (e.g., integers). Implementability in discrete domains is still little understood and has received but little attention in economic literature [7].

We consider the following standard mechanism design setting: There are n players $1 \ldots n$, and a set of outcomes O. Each player i has a private *valuation function* $v_i \in V_i$ that assigns a real value to every $o \in O$ (the higher the value of the outcome the more desirable it is). A (deterministic) social-choice function is a function that assigns an outcome o to every $v \in V$, where V denotes $V_1 \times \ldots \times V_n$. Let V_{-j} denote the cartesian product of all V_is but V_j, and let (v_i, v_{-i}) denote the profile of valuation functions in which player i's valuation function is $v_i \in V_i$, and the other players' valuation functions are as specified by $v_{-i} \in V_{-i}$. Then, f is *implementable* iff there is a payment function p_i such that for every $i \in [n]$, for every $v_{-i} \in V_{-i}$, and for every $v_i, v'_i \in V_i$,

$$v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}).$$

Rochet [11] has shown that any social-choice function is implementable iff a constraint called "*cycle monotonicity*" holds (see recent application by Lavi and Swamy [6]). Bikhchandani et al. [2] proposed the simple *weak-monotonicity* constraint: A social-choice function f is said to be weakly-monotone if for every $i \in [n]$, for every $v_{-i} \in V_{-i}$, and for every $v_i, v'_i \in V_i$, such that $f(v_i, v_{-i}) = o_1$ and $f(v'_i, v_{-i}) = o_2$ it holds that

$$v_i(o_1) + v'_i(o_2) \geq v_i(o_2) + v'_i(o_1).$$

f is said to be *strongly-monotone* if whenever $o_1 \neq o_2$ this inequality is strict [5]. It is easy to show that weak monotonicity is always necessary for the implementability of a social-choice function but that it is not always sufficient.

We start our exploration of incentive-compatibility by focusing on two important types of discrete domains: *Integer grid domains* and *0/1 domains* [13]. Let $V = V_1 \times \ldots \times V_n$ be a domain of valuation functions defined over a set of outcomes O. We can think of every $v_i \in V_i$ as a vector in $R^{|O|}$ specifying a value for every outcome.

DEFINITION 1. *A valuation function domain is an integer grid domain if* $V = Z^{|O|} \times \ldots \times Z^{|O|}$.

That is, an integer grid domain is a domain of valuation functions that can take any combination of integer values. We exhibit an example (due to Lan Yu) that shows that in integer grids weak monotonicity is *not* sufficient to guarantee implementability [13]. In fact, this example can easily be made to hold for any *bounded* integer grid. By bounded integer grid, we simply mean the discrete cube $V = \{0, 1, ..., L\}^{n|O|}$ (for some positive integer $L \geq 1$). In contrast, we show that strong monotonicity is sufficient to obtain implementability in integer grids.

DEFINITION 2. $V = V_1 \times \ldots \times V_n$ *is a 0/1 domain if* $V = \{0,1\}^{|O|} \times \ldots \times \{0,1\}^{|O|}$.

We show that, as in the case of integer grids, in 0/1 domains weak-monotonicity is insufficient for implementability, but strong-monotonicity is.

Open Question 1. *Is strong-monotonicity sufficient for implementability in bounded integer grids?*

When does weak monotonicity guarantee implementability in discrete domains? As we have seen this is not true even in natural discrete settings (like integer grids). In contrast, we present [13] a family of discrete domains in which weak-monotonicity suffices for implementability, which we term *Monge domains*. The proof that this is indeed true for Monge domains takes advantage of the two dimensional version of submodularity (see [3]) that holds for this kind of domains (expressed by Monge matrices, hence the name). Monge domains have a simple combinatorial structure that has many advantages from a mechanism design perspective. We refer the reader to [13] for more details.

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