Solution to Exchanges 7.2 Puzzle: Strategically Choosing Products to Release

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This is a solution to the editors puzzle from Issue 7.2 of SIGecom Exchanges. The puzzle is about predicting what products one can expect to see in a market with given positive and negative synergies between products. The full puzzle can be found online at the SIGecom website [Conitzer 2008].

1. GAME THEORY CRASH COURSE

Before we can derive the solution to the puzzle, we need to review a little game theory. In the puzzle, three firms must each decide (once, independently, simultaneously) what product to release, with the only goal in mind to maximize their respective expected payoffs. This fits the mold of a 3 player strategic form game [von Neumann and Morgenstern 1953]. In a strategic form game, each player, i, has a finite set of *strategies*, S_i, from which one must be chosen by Player i. For the game at hand, we have $S_1 = \{A, B, C\}$, $S_2 = \{D, E, F\}$ and $S_3 = \{G, I, H\}$. Each player, i, has an associated *utility function*, $u_i : \prod_k S_k \to \mathbb{R}$, the value of which that player wants to maximize. The game is played by each player independently choosing an element in their respective strategy sets. After the strategies have been chosen, each player receives the utility associated with the combination of strategies chosen by the players.

A natural extension of the strategy space is to *mixed* strategies, where we allow the players to randomize, with a strategy taking the form of a probability distribution over the *pure* strategy set. Let us denote such a strategy for Player i by p_i, with $p_i(X)$ being the probability of playing X, and let us use P_i to describe the set of mixed strategies for Player i. Let us use $p = (p_1, p_2, p_3)$ to describe a strategy *profile*, with a strategy for each player. The utility functions are extended to these mixed strategies simply by letting them be the expected value under the probability distribution defined by the mixed strategy profile. We only need one last piece of notation before we can solve the game. With X being a strategy for Player i, we let $p|X$ mean the strategy profile p altered so that Player i plays according to X.

There are many different solution concepts for strategic form games, but in this case, we can solve the game completely with a standard tool from game theory,

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namely *iterated strict dominance* (ISD) which we will outline next. A pure strategy $X \in S_i$ for Player i is said to be *strictly* dominated by a mixed strategy $p_i \in P_i$ if, for all strategy profiles $p \in P$ we have that $u_i(p|p_i) > u_i(p|X)$. That is, Player i is always *strictly* better off playing p_i than X, no matter what strategies p the other players play. As the players are assumed to be rational, and therefore habitual utility maximizers, they cannot play strictly dominated strategies, as this would always reduce their utility. As any dominance, together with the players rationality, is common knowledge [Aumann 1976], the players can completely disregard the dominated strategy from the game at hand. Furthermore, the reasoning allows us to iterate the procedure and keep eliminating strictly dominated strategies, with all players realizing the exclusion of the strategies. A thorough treatment can be found in the survey [Brandenburger 1992]. For an even stronger derivation, still only using common knowledge of rationality, please refer to [Pearce 1984].

2. PUZZLE SOLUTION

In the general case, one would have to specify the payoff to all players for all combinations of pure strategies. Luckily, the puzzle at hand gives us the utility functions in a much more compact form:

—A strongly benefits from D and G
 $\Rightarrow u_1(p|A) = 2 \cdot p_2(D) + 2 \cdot p_3(G)$

—B strongly benefits from E and suffers from F
 $\Rightarrow u_1(p|B) = 2 \cdot p_2(E) - 1 \cdot p_2(F)$

—C suffers from E and benefits from F
 $\Rightarrow u_1(p|C) = -1 \cdot p_2(E) + 1 \cdot p_2(F)$

—D strongly benefits from G
 $\Rightarrow u_2(p|D) = 2 \cdot p_3(G)$

—E suffers from H and strongly benefits from I
 $\Rightarrow u_2(p|E) = -1 \cdot p_3(H) + 2 \cdot p_3(I)$

—F benefits from H and suffers from I
 $\Rightarrow u_2(p|F) = 1 \cdot p_3(H) - 1 \cdot p_3(I)$

—G has no relationship to any other product
 $\Rightarrow u_3(p|G) = 0$

—H suffers from A and benefits from B and C
 $\Rightarrow u_3(p|H) = -1 \cdot p_1(A) + 1 \cdot p_1(B) + 1 \cdot p_1(C)$

—I strongly benefits from A and suffers from B and C
 $\Rightarrow u_3(p|I) = 2 \cdot p_1(A) - 1 \cdot p_1(B) - 1 \cdot p_1(C)$

What Player 3 can now observe is that by releasing product H with probability 0.6 and product I with probability 0.4, his ex ante expected payoff is strictly greater than what he would get by releasing product G:

$$\begin{aligned} u_3(p|0.6H + 0.4I) &= 0.6 \cdot u_3(p|H) + 0.4 \cdot u_3(p|I) \\ &= 0.2 \cdot p_1(A) + 0.2 \cdot p_1(B) + 0.2 \cdot p_1(C) \\ &> 0 = u_3(p|G) \end{aligned}$$

The strict inequality is due to the fact that Player 1 has to release one of the products $\{A, B, C\}$.[1] Player 3 can thus conclude, by only examining his own payoffs, that releasing product G is a bad idea, no matter what other products are released. This is common knowledge, and all players can completely disregard the possibility of product G being released, and they can thus remove terms with $p_3(G)$ from their respective utility functions. We now have a smaller game, with S_3 restricted to $\{I, H\}$, that will have the same solution as the original game has.

In this smaller game, Player 2 can identify a strictly dominated strategy of his own; a biased mix of E and F strictly dominates D, since G cannot be played:

$$\begin{aligned} u_2(p|0.4E + 0.6F) &= 0.4 \cdot u_2(p|E) + 0.6 \cdot u_2(p|F) \\ &= 0.2 \cdot p_3(H) + 0.2 \cdot p_3(I) \\ &> \cancel{2 \cdot p_3(G)} = u_2(p|D) \end{aligned}$$

Player 2 therefore cannot consider releasing product D, and we can again reduce the size of the game with $S_2 = \{E, F\}$. This in turn reveals a strict dominance in Player 1's strategies; mix B and C to dominate A:

$$\begin{aligned} u_1(p|0.4B + 0.6C) &= 0.4 \cdot u_1(p|B) + 0.6 \cdot u_1(p|C) \\ &= 0.2 \cdot p_2(E) + 0.2 \cdot p_2(F) \\ &> \cancel{2 \cdot p_2(D)} + \cancel{2 \cdot p_3(G)} = u_2(p|A) \end{aligned}$$

thereby eliminating A, reducing S_1 to $\{B, C\}$. This enables Player 3 to eliminate I, as it is dominated by H:

$$\begin{aligned} u_3(p|H) &= \cancel{-1 \cdot p_1(A)} + 1 \cdot p_1(B) + 1 \cdot p_1(C) \\ &> \cancel{2 \cdot p_1(A)} - 1 \cdot p_1(B) - 1 \cdot p_1(C) = u_3(p|I) \end{aligned}$$

which in turn eliminates E due to domination by F:

$$\begin{aligned} u_2(p|F) &= 1 \cdot p_3(H) \cancel{- 1 \cdot p_3(I)} \\ &> -1 \cdot p_3(H) \cancel{+ 2 \cdot p_3(I)} = u_2(p|E) \end{aligned}$$

Finally, C dominates B:

$$\begin{aligned} u_1(p|C) &= \cancel{-1 \cdot p_2(E)} + 1 \cdot p_2(F) \\ &> \cancel{2 \cdot p_2(E)} - 1 \cdot p_2(F) = u_1(p|B) \end{aligned}$$

We have thereby, using only common knowledge of rationality, reduced all strategy sets to singletons. The only strategy profile left is (C, F, H), which gives a payoff of 1 to all players. The given derivation also proves that there is only one Nash equilibrium, namely being the only surviving strategy profile.

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[1] In all the remaining calculations, the strict inequalities arise because the probabilities on the left hand side form complete probability distributions, and therefore sum to 1.

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