Uncoordinated Two-Sided Matching Markets

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Various economic interactions can be modeled as two-sided matching markets. A central solution concept to these markets are *stable matchings*, introduced by Gale and Shapley. It is well known that stable matchings can be computed in polynomial time, but many real-life markets lack a central authority to match agents. In those markets, matchings are formed by actions of self-interested agents, whose behavior is often modeled by Nash dynamics such as best and better response dynamics. In this note, we summarize recent results on Nash dynamics in two-sided markets.

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1. INTRODUCTION

One main function of many markets is to match agents of different kinds to one another, for example men and women, students and colleges [Gale and Shapley 1962], interns and hospitals [Roth 1984; 1996], and firms and workers. Gale and Shapley [Gale and Shapley 1962] introduced *two-sided markets* to model these problems. A two-sided market consists of two disjoint groups of agents. Each agent has some

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preferences about the agents on the other side and can be matched to one of them. A matching is *stable* if it does not contain a *blocking pair*, that is, a pair of agents from different sides who can deviate from this matching and both benefit. Gale and Shapley [Gale and Shapley 1962] showed that stable matchings always exist and can be computed in polynomial time. Besides their theoretical appeal, two-sided matching models have proved useful in the empirical study of many labor markets such as the National Resident Matching Program (NRMP). They have also been used for designing high school admission processes [Abdulkadiroglu et al. 2005]. Since the seminal work of Gale and Shapley, there has been a significant amount of work in studying two-sided markets, especially on extensions to many-to-one matchings and preference lists with ties [Kelso and Crawford 1982; Roth and Sotomayor 1990; Fleiner 2003; Echenique and Oviedo 2006]. For other examples see the book by Knuth [Knuth 1976], the book by Gusfield and Irving [Gusfield and Irving 1989], or the book by Roth and Sotomayor [Roth and Sotomayor 1990].

In many real-life markets, there is no central authority to match agents, and agents are self-interested entities. This motivates the study of *uncoordinated two-sided markets*, first proposed by Knuth [Knuth 1976]. Uncoordinated two-sided markets can be modeled as a game among agents of one side, which we call the *active* side. The strategy of each active agent is to choose one agent from the *passive* side, and stable matchings correspond to Nash equilibria of the corresponding games. In order to understand the behavior of the agents in these uncoordinated markets, it is interesting to analyze the dynamics that arise when agents play repeatedly better or best responses to the strategies of the other agents. In the next section, we summarize some recent results on these *Nash dynamics*. To capture the idea of lacking coordination, the focus is laid on randomized versions of the dynamics, in which agents improve their strategies in a random order. After that, in Section 3, we discuss extensions to more general models of two-sided markets in which the preference lists of the agents are not strict but can have ties.

2. NASH DYNAMICS

Let us first formalize the terminology. A *two-sided market* consists of two disjoint groups of agents $\mathcal{X}$ and $\mathcal{Y}$, e.g., women and men. Each agent has a preference list over the agents from the other side and can be assigned to at most one them. Given a matching M between $\mathcal{X}$ and $\mathcal{Y}$, we say that two agents $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ form a *blocking pair* if x and y are not matched to each other in M, but x prefers y to her partner in M and y prefers x to his partner in M. We assume that agents prefer every partner to being unmatched. The matching M' is said to be obtained from M by *resolving* the blocking pair (x, y) if the following holds: x and y are matched to each other in M', any partners with whom x and y are matched in M are unmatched in M', and all other edges in M and M' coincide. A matching is *stable* if it does not contain a blocking pair. Stable matchings correspond to the *pure Nash equilibria* when the two-sided market is viewed as a game among the agents. Let M denote a matching between $\mathcal{X}$ and $\mathcal{Y}$, and let $(x, y) \in \mathcal{X} \times \mathcal{Y}$ be a blocking pair in M. We say that agents x and y play a *better response* if the blocking pair (x, y) is resolved. We say that agent x plays a *best response* if there is no blocking pair (x, y') in M such that x prefers y' to y. To understand the behavior of the selfish agents, we study the dynamics that arise when agents repeatedly play better or best responses. In the *better response dynamics*, we start with an arbitrary (possibly partial or even empty) matching M, and in each step one blocking pair is picked and resolved. This process continues until a stable matching is reached. In the *best response dynamics*, we also start with an arbitrary matching, and in each step one agent $x \in \mathcal{X}$ who is involved in at least one blocking pair is picked and the blocking pair (x, y) corresponding to x's best response is resolved.

To model the lack of coordination, we study random versions of these dynamics: In the *random better response dynamics* at each step a blocking pair is chosen uniformly at random among the set of all blocking pairs and resolved. In the *random best response dynamics* at each step an agent from $x \in \mathcal{X}$ is chosen uniformly at random among all agents from $\mathcal{X}$ who are involved in at least one blocking pair and the blocking pair (x, y) corresponding to x's best response is resolved.

2.1 Better Response Dynamics

To the best of our knowledge, Donald Knuth was the first who suggested to consider Nash dynamics in two-sided markets. He showed that the better response dynamics can cycle [Knuth 1976]. This means, it is possible to start with a matching M_1 and to resolve some blocking pairs, leading to a sequence of matchings $M_1, M_2, \ldots, M_k$ with $M_k = M_1$. Hence, in the worst case the better response dynamics never stabilizes. This, however, assumes that blocking pairs are resolved in a certain order, which is not realistic in an uncoordinated environment. Hence, he suggested to analyze the random better response dynamics. Roth and Vande Vate [Roth and Vande Vate 1990] proved that for every matching M, there exists a polynomial sequence of blocking pairs that lead to a stable matching when resolved consecutively. Hence, in two-sided markets all *sink equilibria* are trivial, where a sink equilibrium is a strongly connected component of the state graph without outgoing edges [Goemans et al. 2005]. This result also implies that the random better response dynamics reaches a stable matching in a finite number of steps with probability one. However, it leaves open the question of how long it takes to stabilize. We believe that this is a crucial question as it corresponds to the question of how long an uncoordinated market needs to stabilize. We resolved this question and proved the following theorem.

Theorem 2.1 [Ackermann et al. 2008]. *There exists a family of two-sided markets $I_1, I_2, I_3, \ldots$ with corresponding matchings $M_1, M_2, M_3, \ldots$ such that, for $n \in \mathbb{N}$, I_n consists of n women and n men and the random better response dynamics starting in M_n needs $2^{\Omega(n)}$ steps to reach a stable matching with probability $1 - 2^{-\Omega(n)}$.*

This theorem indicates that coordination is necessary as there exist uncoordinated markets that need with high probability exponential time to stabilize. To prove the theorem, we basically construct for every $n \in \mathbb{N}$ a two-sided market with n agents on both sides with the following two properties: First, the stable matchings have a very special structure, namely in all stable matchings there exists a number $k \in \{1, \ldots, n\}$ such that all agents from $\mathcal{X}$ are matched to their k-th most preferred partner from $\mathcal{Y}$. Second, if in the current matching there exists a k such that at least $m_k \geq 15n/16$ agents from $\mathcal{X}$ are matched to their k-th most preferred partner from $\mathcal{Y}$, then when a random blocking pair is resolved, m_k is more likely to decrease than to increase. This leads to a biased random walk, which takes with high probability an exponential number of steps to reach $m_k = n$.

2.2 Best Response Dynamics

Both Knuth's cycle [Knuth 1976], and Roth and Vande Vate's proof [Roth and Vande Vate 1990] hold only for the better response dynamics, and not for the *best response dynamics*. We extended these results to best responses [Ackermann et al. 2008]. That is, we showed that also the best response dynamics can cycle and that starting from any matching, there exists a short sequence of best responses leading to a stable matching. As a corollary of the proof of the latter result, we obtain that every sequence of best responses starting with the empty matching reaches a stable matching after a polynomial number of steps. Hence, when starting with the empty matching, no central coordination is needed to reach a stable matching quickly if agents play only best responses.

In contrast to this, we showed that Theorem 2.1 can also be extended to the random best response dynamics when arbitrary starting configurations are allowed. Hence, even if agents play only best responses, coordination is necessary if arbitrary initial matchings are allowed.

2.3 Correlated Two-Sided Markets

Now that we have seen that coordination is necessary in general, it is an interesting question whether there exists a non-trivial class of two-sided markets that stabilize quickly even without coordination. One such class are *correlated two-sided markets*, which have received a lot of attention recently [Abraham et al. 2007; Ackermann et al. 2007; Lebedev et al. 2006; Mathieu 2007]. In these markets there is a payoff associated with each pair from $\mathcal{X} \times \mathcal{Y}$ and the preference lists of all agents from $\mathcal{X}$ and $\mathcal{Y}$ are chosen according to these payoffs. That is to say, every agent prefers a partnership with larger payoff to a partnership with smaller payoff.[1]

Lebedev et al. [Lebedev et al. 2006] prove that every correlated market has a unique stable matching, that the best response dynamics cannot cycle, and that from every state a short sequence of best responses leading to a Nash equilibrium exists. They also conclude that the random best response dynamics converges quickly. In contrast to this, Mathieu [Mathieu 2007] presents an exponential lower bound on the convergence time if an adversary selects the next agent to play a best response.

We extend the results for correlated markets to *many-to-one two-sided markets*, in which every agent from $\mathcal{X}$ can be simultaneously matched to several agents from $\mathcal{Y}$ [Ackermann et al. 2008]. We consider the case that the sets to which an agent $x \in \mathcal{X}$ can be matched form a matroid on $\mathcal{Y}$. Such matroid two-sided markets arise naturally if, for example, every employer is interested in hiring a fixed number of workers or if the workers can be partitioned into different classes and a certain number of workers from each class is to be hired. We show essentially that the

[1] In [Lebedev et al. 2006] and [Mathieu 2007] these markets are defined in a different way and they are called *acyclic*. It is, however, shown by Abraham et al. that the classes of acyclic and of correlated markets coincide.

random best response dynamics converges quickly also in correlated matroid two-sided markets.

3. TWO-SIDED MARKETS WITH TIES

In many situations it is not realistic to assume that the preference lists of all agents are strict. In [Ackermann et al. 2007], we introduce a model of *two-sided markets with ties* in which the preference lists of the agents in $\mathcal{Y}$ can have ties. In this model, a decreasing payoff function $p_{x,y} \colon \mathbb{N} \to \mathbb{N}$ is given for every pair $(x, y) \in \mathcal{X} \times \mathcal{Y}$. If multiple agents from $\mathcal{X}$ propose to an agent $y \in \mathcal{Y}$ only the most preferred ones are matched to y. If the most preferred agent is not unique, and $n_y > 1$ agents are matched to $y \in \mathcal{Y}$, then intuitively these agents share the payoff, that is, every agent $x \in \mathcal{X}$ that is matched to y obtains payoff $p_{x,y}(n_y)$. One application of this model are markets into which different companies can invest: as long as the investing companies are of comparable size, they share the payoff of the market, but large companies can utilize their market power to eliminate smaller companies completely from the market.

Interestingly, this model of two-sided markets with ties in not only a generalization of two-sided markets but also of *congestion games*, a well-studied model for resource allocation among uncoordinated agents [Rosenthal 1973]. We prove that two-sided markets with ties possess stable matchings (i.e., pure Nash equilibria), which can be computed in polynomial time. This unifies previous arguments for two-sided markets and congestion games [Ackermann et al. 2007]. Additionally, we consider correlated versions of two-sided markets with ties and show that they are potential games.

4. CONCLUSIONS

We have seen that coordination is necessary in two-sided markets, as otherwise these markets do not stabilize quickly. An interesting question is whether there are other uncoordinated dynamics that find a stable matching quicker. It would also be interesting to see whether there are, besides correlated markets, other non-trivial classes of two-sided markets on which Nash dynamics converge quickly. Finally, let us mention that the instances that we construct in the proof of Theorem 2.1 and its corresponding version for the best response dynamics are very artificial. Studying the convergence time in more realistic input models like, for example, the model of *smoothed analysis* [Spielman and Teng 2004], might lead to valuable insights.

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