Bertrand Competition in Networks

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We briefly summarize our recent work characterizing the efficiency of equilibria in pricing games in two-sided combinatorial markets.

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Imagine a future internet where service providers (ISPs) selfishly price bandwidth to maximize revenue and users route packets along paths that maximize utility (value from routing minus the price paid). How efficiently would such a "bandwidth market" operate? What properties of the market or demand structure affect the efficiency of its equilibrium allocation? In this note we survey our recent work [Chawla and Roughgarden 2008; Chawla and Niu 2009] on characterizing the efficiency of equilibria arising from price competition in such two-sided markets.

We consider revenue-maximizing sellers that own a fixed inventory of a unique item, and consumers that each buy a bundle of items to maximize utility (value minus price). The sellers first set item prices and consumers then buy their favorite bundles. All parties have full information about others' values and constraints. We call this game a *Bertrand game*, following the economics literature on price competition in homogeneous product markets. Bertrand games are a natural model for markets where prices are relatively static and non-discriminatory, and left-over inventory can be disposed off freely.

We focus on Bertrand games in networks where each seller owns a single edge, and each consumer is interested in buying a path from its source vertex to its sink vertex. The efficiency, or social value, of an outcome of the game is the total value obtained by all the consumers from the bought bundles. The *price of anarchy* is the ratio of the optimal efficiency achievable while obeying capacity (supply) constraints to that of the worst Nash equilibrium in the game. The *price of stability* is the analogous ratio with the best Nash equilibrium in the game. These quantities are defined with respect to pure Nash equilibria of the game. We present bounds on them that

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are valid whenever a pure Nash equilibrium exists (sufficiently complex instances do not always have such an equilibrium).

1. THE PRICE OF ANARCHY WITH SINGLE-PARAMETER CONSUMERS

Consider first a "single-parameter" setting in which each consumer values each of their desired bundles (i.e., source-sink paths) equally.

The effect of monopolies. A simple example of a Bertrand game is a homogeneous product market where all items are perfect substitutes and each consumer is interested in buying any one of the items (equivalent to a single-source single-sink parallel-link network). This setting has been studied extensively in the economics literature (see, for example, [Mas-Colell et al. 1995; Harrington 1989; Baye and Morgan 1999] and references therein). It is well known, for example, that in markets with two or more sellers selling homogeneous products with equal marginal costs, the equilibrium price charged by each seller is equal to the marginal cost for the product. That is, the sellers make no profit and the entire social surplus is redistributed back to the consumers. This is known as the Bertrand paradox (see [Wikipedia] for a discussion). Such markets always obtain optimal efficiency. On the other hand, in a market with a single seller (a monopoly), the seller can artificially restrict supply and raise prices to obtain a large profit, hurting market efficiency in the process.

Our first set of results characterizes the efficiency of equilibria in symmetric games (i.e., where all buyers have a common source and sink) as a function of the number of monopolies, where a monopoly is an edge belonging to every source-sink path (i.e., a "cut edge"). Here $\mathcal{L}$ denotes the ratio of the largest consumer value to the smallest.

THEOREM 1.1. (informal, [Chawla and Roughgarden 2008]) *In single-source single-sink Bertrand games with no monopolies, the price of anarchy is* 1*. With one monopoly, the prices of anarchy and stability can be as large as* $\log \mathcal{L}$ *but no larger. When the number of monopolies is* $k > 1$*, the price of anarchy cannot be bounded in terms of* $\mathcal{L}$*. The largest-possible price of stability is* $O(\mathcal{L}^{k-1})$*, and this bound is tight up to a constant factor.*

Multiple sources and sinks. In more general networks, equilibria can be inefficient even in the absence of monopolies. This occurs because congestion in some parts of the network can create "virtual" monopolies in other parts of the network. Our second result bounds the price of anarchy in networks with multiple source vertices in which all consumers have the same sink vertex and the distribution of consumer values is the same at each source vertex. A key parameter turns out to be the *sparsity* $\alpha \leq 1$ of the network, defined as the largest fraction of the consumers that can be routed simultaneously from every source to the sink while obeying capacities on the links. We then have the following result relating the price of anarchy in the Bertrand game to the sparsity of the underlying network.

THEOREM 1.2. (informal, [Chawla and Niu 2009]) *In every single-sink instance of the Bertrand game that admits an equilibrium, contains no monopolies, and has the same distribution of values at each source, the price of anarchy is no more than* $1/\alpha$*, where* α *is the sparsity of the underlying network.*

The bound in this theorem is tight and all of the assumptions are necessary. In particular, there are networks with two sources and one sink where the price of anarchy is arbitrarily close to $1/\alpha$. Furthermore, the theorem does not extend to multiple-source multiple-sink networks.

THEOREM 1.3. (informal, [Chawla and Niu 2009]) *There exists a multiple-source multiple-sink network with no monopolies such that for* every *"non-trivial" demand distribution, the price of anarchy of the Bertrand game on the network is unbounded.*

Value distributions with a monotone hazard rate. Our bounds improve if we impose additional assumptions on the distribution of consumer values. Specifically, we study distributions satisfying the "monotone hazard rate" condition and show that if the value distribution at every source in the network is the same and satisfies this condition, then the price of anarchy is no more than e^k, where k is the length of the longest source-sink path in the network [Chawla and Niu 2009]. This condition is common in the mechanism design literature and is satisfied, for example, by the uniform, normal, exponential, power-law (for exponents greater than one), Laplace, and chi-square distributions [Bagnoli and Bergstrom 2005].

2. GENERAL CONSUMER VALUATIONS

Finally, in [Chawla and Niu 2009] we investigate a more general version of Bertrand games where consumers value different bundles at different amounts. In the network setting, the difference in values for different paths can arise due to differences in, for example, latencies, packet losses, jitter, etc. We consider two different models for consumer values. In the related values model, each consumer u has an intrinsic value ℓ_u and each path p is associated with a quality Q_p. The value of consumer u for path p is simply $\ell_u + Q_p$. In the unrelated values model consumer values are arbitrary. We study these models in single-source single-sink parallel link networks. In the absence of monopolies (defined appropriately), the behavior of the related values model is identical to the single-parameter valuations setting: the price of anarchy is 1. In the unrelated-values version, the behavior of the game is very different. Pure equilibria need not exist. Also, even when there are no monopolies in the network, some edges can effectively become monopolies when consumers prefer one edge much more than the others, which can result in a large price of anarchy. Nevertheless, we prove that the price of anarchy is always bounded above by a factor logarithmic in $\mathcal{L}$.

3. RELATED WORK

A number of recent papers, surveyed in [Ozdaglar and Srikant 2007], study the inefficiency of equilibria in network pricing games. Most of these works assume that the consumer faces two kinds of costs for routing its traffic—the prices charged by the edges, and a latency cost owing to other traffic on the path. In our model, by contrast, each edge has a hard capacity constraint rather than a congestion-dependent cost function. While users do not face congestion costs per se in our model, service providers are incentivized to keep usage on links below their capacities, and to pass on the "costs" associated with oversaturated edges to consumers in the form of higher usage prices.

4. OPEN QUESTIONS

We provided a number of tight bounds on the price of anarchy in Bertrand games in networks with single-parameter consumers as a function of the variation in consumer valuations and desired bundles, the congestion in the network (i.e., the sparsity), the shape of the demand distributions, etc. The most interesting open question related to our work is to study equilibria in reasonably general networks when consumers can have different values for different bundles. Another potentially interesting direction is to characterize the Bertrand games that admit (pure) Nash equilibria.

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