Bisection Auctions

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In this note we give a survey of bisection auctions. Bisection auctions have been introduced in order to reduce the number of rounds and increase privacy of information in iterative implementations of Vickrey auctions. First, we present the case of discrete valuations. we discuss the strategic properties of this auction and recent results which show that—for 2 bidders—the auction dominates in a particular sense any other auction with respect to the number of bits revealed. For the case of continuous valuations we contrast its properties with the result that no practical query auction can achieve full efficiency in ex-post equilibrium.

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1. INTRODUCTION

In [Grigorieva et al. 2007], we introduced a single-item auction that is based on binary search or *bisection*. Our initial goal was designing an iterative auction that implements the Vickrey auction as an iterative auction, like the English auction, but with a small number of rounds. Independently, [Fujishima et al. 1999] had proposed to use bisection in single item auctions, but did not analyze the game theoretic properties. Since then, we have published a series of papers on this auction and related issues. The auction has been used in [Fadel and Segal 2009] as a tool to analyze communication costs of mechanisms. And [Feigenbaum et al. 2009] have recently studied variants of the bisection auction in terms of privacy approximation. We want to summarize here our main findings on this auction and its offspring, the family of c-fraction auctions. We start with the case of integer valuations, and then explain how we adjusted the bisection auction to the continuous setting.

2. DISCRETE TYPES

In the discrete setting let us assume that private valuations are integers in the interval $[0, 2^R)$ for some $R > 1$. Assume we have N bidders. The goal of the bisection auction can be stated in two ways: (1) implement the Vickrey auction as an iterative auction with a small number of rounds, (2) implement the Vickrey auction by revealing as few bits of the binary encoding of the bidders valuations as

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possible. Let me first explain how the bisection auction proceeds, give an example and finally summarize its properties.

The bisection auction has R rounds. The price sequence starts at the middle of the initial interval with a price equal to 2^{R-1}. (Instead of choosing the middle of the interval one may choose any integer inside the interval. In particular, in case of not uniformly distributed valuations one should choose the median in order to let the analysis for the number of rounds go through.) Bidders report their demand at the current price by sealed bids. A *yes*-bid stands for the announcement to be willing to buy at the current price, a *no*-bid for the contrary. As a function of these bids, the auctioneer announces the price of the next round.

In case there are at least two players submitting a *yes*-bid, the price goes up to the middle of the upper half interval, i.e., the interval $[2^{R-1}, 2^R)$. The players that are allowed to participate actively in the next round are the ones that said *yes* and they are competing for the object in the price range $[2^{R-1}, 2^R)$. The other players drop out of the auction and no longer have any influence on the proceedings of the auction. In case there is at most one player saying *yes*, attention shifts to the lower half interval, i.e. the interval $[0, 2^{R-1})$ and the price goes down to the middle of this interval. Two different things can happen now. First, the easy case, if no-one has submitted a *yes*-bid. In that case all active players remain active in the next round. In the other case there is a single player that submitted a *yes*-bid. This player now becomes the winner and he gets the object. Nevertheless the auction doesn't end, but enters a price-determination phase. The active players in the next round are the ones that were active in the previous round minus the winner. In order to keep active players motivated to participate in the auction they should not get to know that the object has already been assigned. Therefore we assume that bidders aren't able to observe bids of the others. The remaining active players are competing on the lower half interval $[0, 2^{R-1})$. The winner, although he is no longer considered to be active, is considered to say *yes* to all prices that are proposed beyond the moment he became the winner. After all, all these prices will be lower than the price he agreed to when he became the winner. Apart from this, the way it is decided whether the price should go up or down is not any different from the way this is decided in the winner-determination phase. In each round depending on submitted bids we subsequently restrict attention to either the lower half of the current interval, or to the upper half of the current interval.

Iterating this procedure will eventually yield a winner and a price. In the case when in no round precisely one player said *yes*, several players will still be active after R rounds, and the object is assigned by a lottery to one of them. The price is uniquely determined because in each round the length of the current interval goes down by a factor of two. Since the initial interval is of length 2^R, after R rounds the resulting interval is of length 1. And since it is a half-open interval, it contains exactly one integer. This integer is declared to be the price the winner of the auction has to pay for the object.

Example. This example illustrates how the bisection auction works. Suppose there are four bidders, A, B, C, and D, with the following integer private valuations from the interval $[0, 16)$: 11, 7, 15, 9. To determine the winner and the price in this setting the bisection auction takes four rounds and starts with an ask price equal to 8. Suppose that each bidder chooses to respond truthfully and follows a straightforward strategy under which he says *yes* if an ask price is less than or equal to his valuation and *no* otherwise. Bidders are not informed about other bidders' choices. The bisection auction proceeds as follows:

round	price	lower bound	upper bound	A	B	C	D
1	8	0	16	yes	no	yes	yes
2	12	8	16	no	-	yes	no
3	10	8	12	yes	-	(yes)	no
4	11	10	12	yes	-	(yes)	-

Since three bidders submitted *yes*-bids in the first round, the price increases to the middle of the current price and the current upper bound. So the ask price of the second round is 12. These three bidders remain active while bidder B drops out. Since there is only one *yes*-bid in the second round we have a winner and we enter what we call the price determination phase. From now on, the winner, bidder C, is considered to say *yes*. Players A and D are still active. In the third round, there are two *yes*-bids so the price increases. Player D drops out. In the fourth round, the auction terminates. Taking into account bids made during the last round we compute the final lower and upper bounds. Since there were 2 *yes*-bids the upper bound remains 12 while the lower bound becomes 11. The winner, bidder C, takes the object and pays price 11 which is the second highest value of the bidders that participated in this auction.

We have shown in [Grigorieva et al. 2007] that answering each query in the bisection auction truthfully is a dominant strategy. Showing this is easy if bidding strategies are restricted to so-called threshold strategies. In such a strategy each bidder selects a threshold and uses this to answer the query. In the bisection auction the highest threshold will win, and the winner will pay an amount equal to the second highest threshold. By the usual Vickrey argument, choosing a threshold equal to your valuation is a dominant strategy. The difficult part in the proof is to show that one can restrict oneself to threshold strategies. This requires to show that any strategy in the extensive form game defined by the bisection auction is outcome equivalent to a threshold strategy.

In [Grigorieva et al. 2006a] we have analyzed the bisection auction in terms of the expected amount of private information that is revealed. We prove, under the assumption of uniform distribution of types, an exact recursive formula for the number of bits of bidders' valuations that are transferred (note that a truthful answer to any query in the auction reveals exactly one bit). Based on the formula we show that the average number of bits communicated is bounded by $2n+R$. This means that–except for the R bits of the second highest valuation–on average 2 bits per bidder are revealed, independent of the size of R.

This issues the question whether one can do any better. Note that in the worst case, any Vickrey auction has to be informed about all bits from the bidders' valuations. Therefore, we cannot differentiate Vickrey auctions by this worst case measure. We need a finer measure of comparison. In [Grigorieva et al. 2009] we introduce the following measure. Take any implementation of a Vickrey auction that iteratively queries bidders for some bit from their valuations. Then let for every $k < N2^{R-1}$, $F(A, k)$ be the number of instances for which some auction A ends after at most k queries. We say that an auction A dominates an auction B if $F(A, k) \geq F(B, k)$ for all k. Note that for the uniform distribution this implies that the average number of bits communicated in A is smaller than in B. The main result that we could achieve says that for 2 bidders, the bisection auction dominates every other implementation of the Vickrey auction. For three bidders, one will need to modify the bisection auction to make this true. But it remains a conjecture whether the modified bisection auction is indeed optimal.

Recently, [Feigenbaum et al. 2009] showed that the bisection auction performs also well with respect to a different measure of privacy preservation.

3. CONTINUOUS TYPES

Let us now assume that types are continuous from some interval $[l_0, u_0]$. Obviously searching for the second highest valuation is now not possible anymore. Rather we have to stop the auction at some earlier point. The modification that we have proposed in [Grigorieva et al. 2006b] is the following.

We distinguish in each round r between a price p_r and a query price q_r. In the beginning, $p_0 = l_0$ and $q_0 = (l_0 + u_0)/2$. In each round r the set of active bidders is denoted by A_r. All bidders are active, that is $A_0 = N$. In each round r we ask all active bidders if they are willing to pay price q_r. If no bidder is willing to pay q_r, we set $q_{r+1} = (p_r + q_r)/2$ and $u_r = q_r$. If two or more bidders are willing to pay q_r, we set $p_{r+1} = q_r$, and $q_r = (u_r + q_r)/2$. If exactly one bidder is willing to pay q_r, he is the winner and pays p_r. The auction stops immediately. Since this event may never happen, we introduce a rule how the item is allocated if the auction "does not end". In the beginning of the auction we rank the bidders randomly, and announce the ranking. In case the auction "does not end" the bidder with the highest rank will win. (It turns out that in equilibrium this rule will never have to be activated, which means that we do not have to make precise what it means that the auction "does not end".) In contrast to the bisection auction for discrete types, all bids are now public, and bidders are queried by increasing rank.

It turns out that this auction has an ex-post equilibrium. Intuitively, there is sometimes a safe bet for a bidder that has a value in $[p_r, q_r)$ that lets him say *yes* despite the risk that the price may increase to q_r. Indeed, suppose in such a round r all of the still active bidders who are asked before him say *no*. If he says yes, two things can happen. Either nobody else after him says yes, in which case he wins at price p_r. Or, one bidder after him says *yes* as well. Now the price becomes indeed higher than his valuation, but because of the ranking he can protect himself against winning by saying no in all following rounds. In the worst case all other active bidders will do the same. But then he will be protected by the tie breaking rule. Since in this strategy a bidder is most of the time truthful, except for this one occasion, we have called it the *bluff* strategy.

The profile of bluff strategies forms an ex-post equilibrium. But it has not only nice game theoretic properties. Firstly, observe that the option that prices could drop is never used. Second, the price increases rapidly. By not subdividing the current interval into halves, but setting the query price at $c(p_r + q_r)$ for some c we can even guarantee that the probability of an inefficient allocation as well as the level of inefficiency becomes arbitrary small, though at the cost of slightly increasing running time. This works for any distribution of types: we set in each round the query price q_r such that the conditional probability of a bidder having a valuation between (p_r, q_r), given that his valuation is in (p_r, u_r), equals c.

The c-fraction auction provides a design which allows the auctioneer to trade off inefficiency with running time. Thereby, the auction terminates always after a finite number of rounds. In [Grigorieva et al. 2009] we show that this is in a sense all we can hope for in an individually rational auction with an ex-post equilibrium. Indeed, we prove that an individual rational query auction that is always computes the efficient allocation has infinite running time with probability 1.

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