Convergence to Equilibrium in Local Interaction Games

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We study a simple game-theoretic model for the spread of an innovation in a network. The diffusion of the innovation is modeled as the dynamics of a coordination game in which the adoption of a common strategy between players has a higher payoff. Classical results in game theory provide a simple condition for the innovation to spread through the network. The present paper characterizes the rate of convergence as a function of graph structure. In particular, we derive a dichotomy between well-connected (e.g. random) graphs that show slow convergence and poorly connected, low dimensional graphs that show fast convergence.

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1. INTRODUCTION

How does a new behavior, a new technology or a new product diffuse through a social network? The computer science literature has addressed this question by studying epidemic or cascade models (see for instance, Kleinberg's survey [Klein07]). In these models, the underlying assumption is that people adopt an innovation when they come in contact with others who have already adopted, that is, innovations spread much like epidemics.

The present paper studies a different class of models, that has been originally proposed within evolutionary game theory. The basic hypothesis here is that, when adopting a new behavior, each individual makes a rational choice to maximize his or her payoff in a game. This is more suitable for modeling the diffusion phenomena in scenarios where the individuals' behavior is the result of a strategic choice among competing alternatives.

The social network is represented by a graph $G = (V, E)$ where each vertex represents an individual. The current strategy (or behavior) adopted at vertex $i \in V$ is described by a variable $x_i \in \{+1, -1\}$. The strategy at i is revised at the arrival times of a Poisson clock with rate one. The new strategy is chosen according

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to the logit distribution [Blu93]:

$$p_{i,\beta}(y_i|\underline{x}_{N(i)}) \propto \exp\left\{\beta \sum_{j\in N(i)} u_{ij}(y_i, x_j)\right\}. \quad (1)$$

Here $N(i)$ is the set of neighbors of i and $u_{ij}(y_i, x_j)$ is the utility function of a symmetric 2×2 game. We assume all the games to be identical coordination games: $u_{ij}(y_i, x_j) = u(y, x)$ with $u(+,+) > u(-,+)$ and $u(-,-) > u(+,-)$.

The parameter β determines the amount of 'noisiness' of the dynamics. The limit $\beta = \infty$ corresponds to 'best-response' dynamics: each player adopts the strategy that maximizes its utility. The fixed points of this dynamics coincide with Nash equilibria, whose number can grow exponentially in the number of vertices. In an influential paper, Kandori, Mailath and Rob [KMR93] pointed out that the noisy best-response dynamics has a unique stationary distribution (being irreducible and aperiodic) and that as $\beta \to \infty$ the distribution converges to one of the homogeneous equilibria: either all $+1$, or all -1. Which of these equilibria is selected can be easily determined from the payoff matrix $u(\,\cdot\,,\,\cdot\,)$: this criterion is called 'risk dominance'. The risk dominant strategy is not the one yielding the highest payoff if played by everybody ('payoff dominant') but rather is the one yielding the highest payoff when the adversaries play uniformly at random.

To be definite, let us assume that the risk-dominant strategy is $+1$. By the above remarks, in the long run almost everybody will play that strategy if β is large enough. This is therefore a promising model to describe the diffusion of a new behavior, with $+1$ and -1 corresponding (respectively) to the new and old behavior. In this paper we consider two fundamental questions

(a) How quickly does the $+1$ strategy spread through the network? How does the convergence time depend on the network structure?

(b) What does a typical trajectory of the diffusion process look like?

Ellison [Ell93] was the first to investigate the convergence time and its dependence on the graph structure. He used ideas from Friedlin-Wentzell theory of randomly perturbed dynamical systems that are still the standard tools in the game theory literature. With this technique, he was able to characterize convergence times in complete graphs and one-dimensional graphs.

2. MAIN RESULT

We build upon and extend the 'modern' Markov chain theory and the relationship between noisy best-response logit dynamics and Glauber dynamics for the Ising model [BK+05]. In fact, some of the techniques developed in the current paper are of independent interest in that literature.

The analysis enables us to bound the rate of convergence in terms of tilted-cutwidth of the graph. We refer the reader to the full version of the paper [MS09] for the definition and the detailed analysis. Instead, we state the main implications of our result on the rate of convergence for a few classic models of social networks.

To this end, we let T_+ denote the hitting time or convergence time to the all-$(+1)$

configuration, and define the *typical hitting time* for $\underline{+1}$ as

$$\tau_+(G;\underline{h}) = \sup_{\underline{x}} \inf \left\{ t \geq 0 : \mathbb{P}_\beta^{\underline{x}}\{T_+ \geq t\} \leq e^{-1} \right\} . \tag{2}$$

Let us also define a few familiar and natural models of social networks:

(a) *Random graphs.* This class includes random regular graphs of degree $k \geq 3$, random graphs with a fixed degree sequence with minimum degree 3, and random graphs in preferential-attachment model with minimum degree 2 [MPS06; GMS03].

(b) *d-dimensional networks.* We say that the graph G is *embeddable* in d dimensions or is a d-dimensional range-K graph if one can associate to each of its vertices $i \in V$ a position $\xi_i \in \mathbb{R}^d$ such that, (1) $(i,j) \in E$ implies $d_{\rm Eucl}(\xi_i, \xi_j) \leq K$ (here $d_{\rm Eucl}(\cdots)$ denotes Euclidean distance); (2) Any cube of volume v contains at most $2\,v$ vertices.

(c) *Small-world networks.* The vertices of this graph are those of a d-dimensional grid of side $n^{1/d}$. Two vertices i, j are connected by an edge if they are nearest neighbors. Further, each vertex i is connected to k other vertices $j(1), \ldots, j(k)$ drawn independently with distribution $P_i(j) = C(n)|i-j|^{-r}$.

THEOREM 2.1. *As $\beta \to \infty$, the convergence time is $\tau_+(G) = \exp\{2\beta\Gamma_*(G) + o(\beta)\}$ where*

(*i*) *If G is a random k-regular graph with $k \geq 3$, a random graph with a fixed degree sequence with minimum degree 3 or a preferential-attachment graph with minimum degree 2, then for h small enough, $\Gamma_*(G) = \Omega(n)$.*

(*ii*) *If G is a d-dimensional graph with bounded range, then for all $h > 0$, $\Gamma_*(G) = O(1)$.*

(*iii*) *If G is a small world network with $r \geq d$, and h is such that $\max_i h_i \leq k - d - 5/2$, then with high probability $\Gamma_*(G) = \Omega(\log n / \log\log n)$.*

(*iv*) *If G is a small world network with $r < d$, and h is small enough, then with high probability $\Gamma_*(G) = \Omega(n)$.*

The basic implication of the above theorem is that if the underlying interaction or social network is well-connected, i.e. if it resembles a random-regular graph or a power-law graph, then the +1 action spreads very slowly in the network. On the other hand, if the interaction is restricted only to individuals that are geographically close, then convergence to +1 equilibrium is very fast.

3. CONCLUSION

These examples highlight an important difference between game theoretic and epidemic models. In epidemic models, the innovation spreads very quickly in well-connected networks. Moreover, high degree nodes expedite the rate of diffusion significantly. Our analysis on coordination games predicts strikingly different behaviors and therefore it challenges the common assumption that the diffusion of viruses, new technologies, and new political or social beliefs all have the same "viral" behavior. A deeper understanding of these differences will be critical for making predictions about these dynamics or developing algorithms for spreading or containing them.

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