Solution to Exchanges 7.3 Puzzle: Product Adoption in a Social Network

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Two correct solutions were submitted to the puzzle in SIGecom exchanges given at `http://www.sigecom.org/exchanges/volume_7/3/PUZZLE.pdf`. Both of these solutions are listed below. The first is by Aneesh Sharma, the second by Sicco Verwer.

Solution 1

Let B_i denote the expected number of B adoptors after i agents have made their adoption decisions. We are interested in computing ${}^{B_n}/_{n}$. First, we observe that $B_1 = p_0$ as the first agent can only adopt B if she is a B fanatic. Further, we observe that for any $i > 1$:

$$B_{i+1} = \left(p_0 + \frac{B_i}{n}p_1\right)(B_i + 1) + \left(1 - p_0 - \frac{B_i}{n}p_1\right)B_i$$

This is because the expected number of agents go up by 1 only if either agent $i+1$ is a B fanatic or if the agent that $i+1$ has chosen to admire has already chosen to adopt B (with probability ${}^{B_i}/_{n}$). In the remaining cases, the expected number of agents remain the same. Now, we can simplify the above equation to get:

$$B_{i+1} = p_0 + \left(1 + \frac{p_1}{n}\right)B_i$$

We can telescope this sum starting with B_1 to get:

$$B_{i+1} = p_0\left(1 + \left(1 + \frac{p_1}{n}\right) + \ldots + \left(1 + \frac{p_1}{n}\right)^i\right)$$

Summing the series for $i = n-1$ and using the approximation $(1 + {}^{x}/_{n})^n \approx e^x$ for large n, we have the quantity of interest as:

$$\frac{B_n}{n} \approx \frac{p_0}{p_1}\left(e^{p_1} - 1\right)$$

Solution 2

Like the previous solution, we obtain:

$$B_{i+1} = p_0 + \left(1 + \frac{p_1}{n}\right)B_i$$

However, we write out this sum as:

$$B_n = p_0\left(n + a_n \cdot \frac{p_1}{n} + b_n\left(\frac{p_1}{n}\right)^2 + c_n\left(\frac{p_1}{n}\right)^3 \ldots\right)$$

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Now we only need to find the values for $a_n, b_n, c_n, \ldots$. Here a_n stands for the number of times that $p_0 \cdot {}^{p_0}/_n$ occurs in B_n. This is equal to the number of times that $p_0 \cdot {}^{p_0}/_n$ occurs in B_{n-1} plus the number of times that p_0 occurs in B_{n-1}. Thus:

$$a_n = a_{n-1} + (n-1)$$

Similarly for $b_n, c_n, \ldots$:

$$\begin{aligned} b_n &= b_{n-1} + a_{n-1} \\ c_n &= c_{n-1} + b_{n-1} \\ &\ldots \end{aligned}$$

Except the first few values, these recursions form the triangular, tetrahedral, pentopic, etc. numbers. Solving these recursions results in the following sets of equations:

$$\begin{aligned} a_n &= \frac{1}{2} n (n+1) \\ b_n &= \frac{1}{6} n (n+1)(n+2) \\ c_n &= \frac{1}{24} n (n+1)(n+2)(n+3) \\ \ldots \end{aligned}$$

For n goes to ∞, these numbers can be used to rewrite the final result ${}^{B_n}/_n$ as:

$$\frac{B_n}{n} \approx p_0 \left(1 + \frac{p_1}{2} + \frac{p_1^2}{6} + \frac{p_1^3}{24} \ldots\right)$$

The sequence $f(x) = 2, 6, 24, \ldots$ is equal to $f(x) = (x+2)!$, thus:

$$\frac{B_n}{n} \approx p_0 \left(1 + \sum_{i=2}^{\infty} \left(\frac{p_1^{i-1}}{i!}\right)\right)$$

Using $e^x = \sum_{i=0}^{\infty} ({}^{x^n}/_{n!})$, we obtain:

$$\begin{aligned} \frac{B_n}{n} &\approx p_0 \left(1 + \frac{1}{p_1} \sum_{i=2}^{\infty} \left(\frac{p_1^i}{i!}\right)\right) \\ \frac{B_n}{n} &\approx p_0 \left(1 + \frac{1}{p_1} \left(\sum_{i=0}^{\infty} \left(\frac{p_1^i}{i!}\right) - 1 - p_1\right)\right) \\ \frac{B_n}{n} &\approx p_0 \left(1 + \frac{1}{p_1} \left(e^{p_1} - 1 - p_1\right)\right) \\ \frac{B_n}{n} &\approx \frac{p_0}{p_1} \left(e^{p_1} - 1\right) \end{aligned}$$