Geometry and Auctions

Paul Klemperer, February 2021

My recent papers are all with Elizabeth Baldwin
We have also written with
Martin Bichler, Max Fichtl, Paul Goldberg, and Edwin Lock

Using geometry to represent bidders' preferences in auctions

--construct (new) **bidding languages** from simple pieces
--aggregating the pieces give wide classes of preferences
--easy to understand and analyse

--also helps understand competitive equilibrium for indivisible goods
(see Baldwin & Klemperer, *Econometrica*, 2019)

My recent papers are all with **Elizabeth Baldwin**
We have also written with
Martin Bichler, Max Fichtl, Paul Goldberg, and Edwin Lock
on solving these (new) auctions - i.e., finding prices & allocations

Northern Rock Bank Run - September 2007

(1 year before Lehman Brothers & Global Financial Crisis)

→ Bank of England wanted to “sell” multiple “types” of loans to commercial banks, building societies, etc.,
“type” = quality of collateral used by borrower

Why not just run a separate auction for each “variety”?

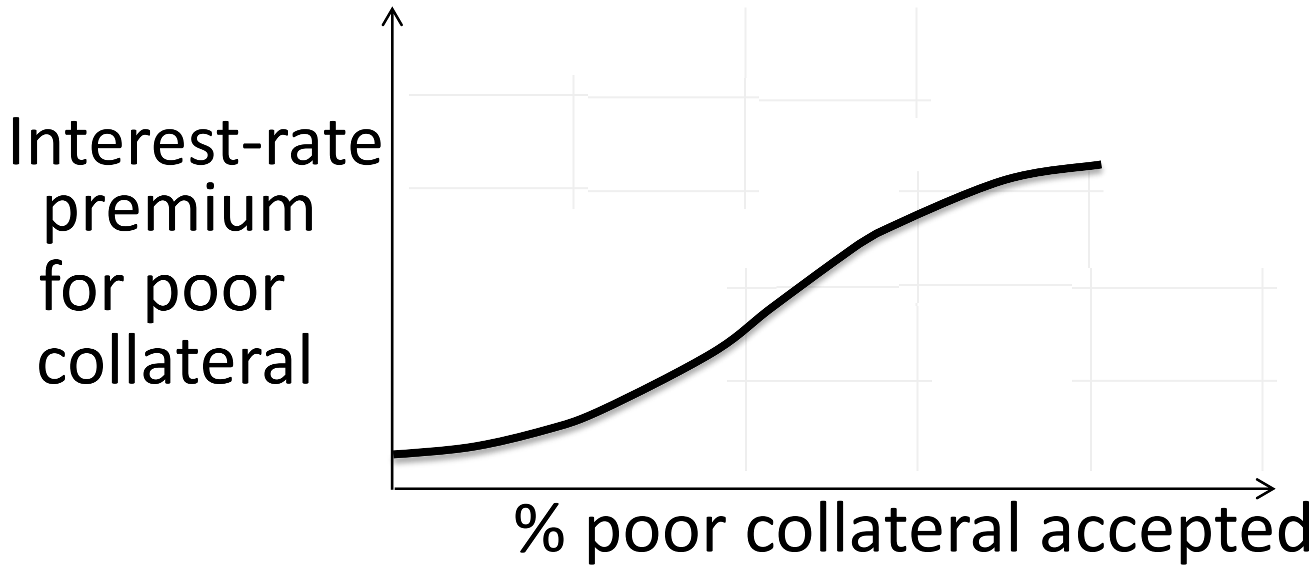
(1) *Market power:* too little competition in each auction

(2) *Bidders have to decide: which auction to enter?*

e.g., potential borrower (commercial bank) might prefer to say:
"will use my poor collateral if price difference < (e.g.)15 basis pts”

(3) *Auctioneer has to decide: how much to sell in each auction?*

e.g., Bank of England might prefer this (if just two qualities):



Why not run Simultaneous Multiple Round Auctions?

as pioneered by Paul Milgrom and Bob Wilson,
e.g., I ran SMRA for UK's 3G mobile-phone licences (with Ken Binmore)
(2 large + 3 small licences; raised $34 billion = 2½% of GNP)

BUT ***-- may take too long***

-- may aid collusion &/or predation

-- hard to allow the mix of varieties sold to depend upon the bids

→develop **"Product-Mix Auctions"** (Klemperer, 2008)

to sell multiple varieties of a **Product**
when auctioneer's costs depend on **Mix** of varieties sold

ADVERT: Free, open-source software
developed by Elizabeth Baldwin and Paul Klemperer with valuable help from Simon Finster
for several versions of Product-Mix Auctions
is at pma.nuff.ox.ac.uk (& via Paul Klemperer's website)

Product-Mix Auctions

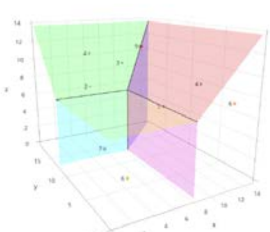
We provide open-source software for three different Product-Mix Auctions. (All Product-Mix Auctions are single-round sealed-bid auctions for multiple units of multiple distinct goods.)

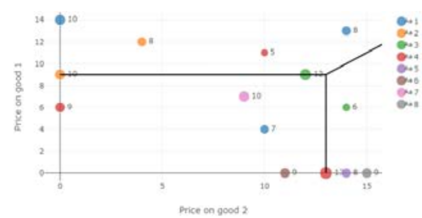
- **Standard Version.**
  - This is a generalisation of the design that is the basis of the "ILTR" auction that is now regularly used by the Bank of England — see paper.
  - It permits flexible buyer and seller preferences over an arbitrary number of goods.
- **Version for Budget-constrained bidders.**
  - This version was originally designed for the Government of Iceland.
  - As in the standard version, bidders simultaneously make sets of bids. In this case, however, each of a bidder's bids can specify a total budget to be spent.
- **"Positive and negative dot-bids" Version.**
  - This version extends the standard version to permit all participants (buyers and seller) to express *any* strong substitute preferences for indivisible goods.

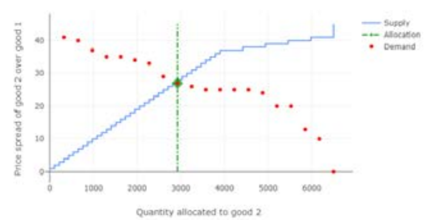
This software is available both as a command-line program `pma`, and a web application `pma-server` (all open-source). The latter provides a single-user interface allowing an auction specification to be constructed in a web browser form. We host the web application here and also provide the code so users can host it locally.

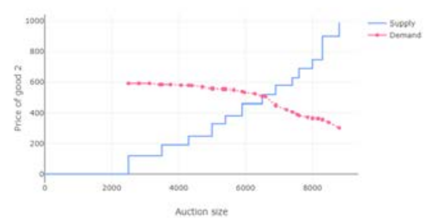
Please contact Elizabeth Baldwin or Paul Klemperer for the password to access this software, or for further information about it; its use is free of charge.

Download software and documentation (requires password)









Product-Mix Auctions

1. Bidders simultaneously state their preferences (as "bids"); alternative ways to express these, depending on context.

*Bids can be understood, explained, and analysed **geometrically**.*

2. Implement Competitive Equilibrium allocation consistent with stated preferences ("bids")

Basic version uses uniform prices

[=(lowest) competitive equilibrium prices]

“Proxy” Mechanisms

Dynamic (Multi-Round) **Auction**	**Static** (Single-Round) **“Proxy” Auction**
Single-unit ascending	Sealed-bid 2nd-price
Multi-unit ascending (homogeneous units)	Uniform-price
Multi-unit **Multi-variety** ascending (Simultaneous Multiple-Round Auction)	***Simplified version of*** **Bank of England Product-Mix Auction***

*Full PMA allows auctioneer to express preferences about how quantities bought/sold depend on bidding

Product-Mix Auctions

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Basic version uses uniform prices
[=(lowest) competitive equilibrium prices]

Auction will work well if ***bidders*** bid (approx.) true values

--auctioneer need ***not*** state her actual preferences
(Bank of England does in its implementation;
Icelandic government planned not to in its)

Product-Mix Auctions

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Bids can be understood, explained, and analysed ***geometrically****.*

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Basic version uses uniform prices
[=(lowest) competitive equilibrium prices]

→ Bidding is *efficient, informative, and easy*
IF bidders can communicate their preferences easily and accurately, which they can …

Product-Mix Auctions

1. Bidders simultaneously state their preferences (as "bids"); alternative ways to express these, depending on context.

*Bids can be understood, explained, and analysed **geometrically**.*

2. Implement Competitive Equilibrium allocation consistent with stated preferences ("bids")

Geometric Language is:

--easy for bidders to understand, and use,
and bidders can express their preferences accurately

--easy for auctioneer to understand, and trust,
and auctioneer can find competitive equilibrium

--give bidders incentive to state their true preferences
(or near enough), and is robust against other manipulation

Bank of England's Product-Mix Auction (1st version)

Two types of goods auctioned by Bank of England

(i) 3-month loans against **Type A** ("weak") collateral
e.g., mortgage-backed securities

(ii) 3-month loans against **Type B** ("strong") collateral
e.g., UK government bonds

Bidders (commercial banks, etc.) can bid for one or both goods, and/or express preferences between them

Total allocation=£5 billion

Bank of England's Product-Mix Auction Now extended

Two types of goods auctioned by Bank of England

Now more goods

(i) 3-month loans against **Type A** ("weak") collateral
e.g., mortgage-backed securities

(ii) 3-month loans against **Type B** ("strong") collateral
e.g., UK government bonds

Bidders (commercial banks, etc.) can bid for one or both goods, and/or express preferences between them

& similar auctions for loans of other durations

Total allocation=£5 billion **now flexible amount of £**

Bank of England's Product-Mix Auction (1st version)

Two types of goods auctioned by Bank of England

(i) 3-month loans against **Type A** ("weak") collateral
e.g., mortgage-backed securities

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Bidders (commercial banks, etc.) can bid for one or both goods, and/or express preferences between them

Total allocation=£5 billion

Example of a bid for (at most) a single piece of fruit: this bid says “willing to pay up to 60¢ for an Apple, I have no interest in a Banana”

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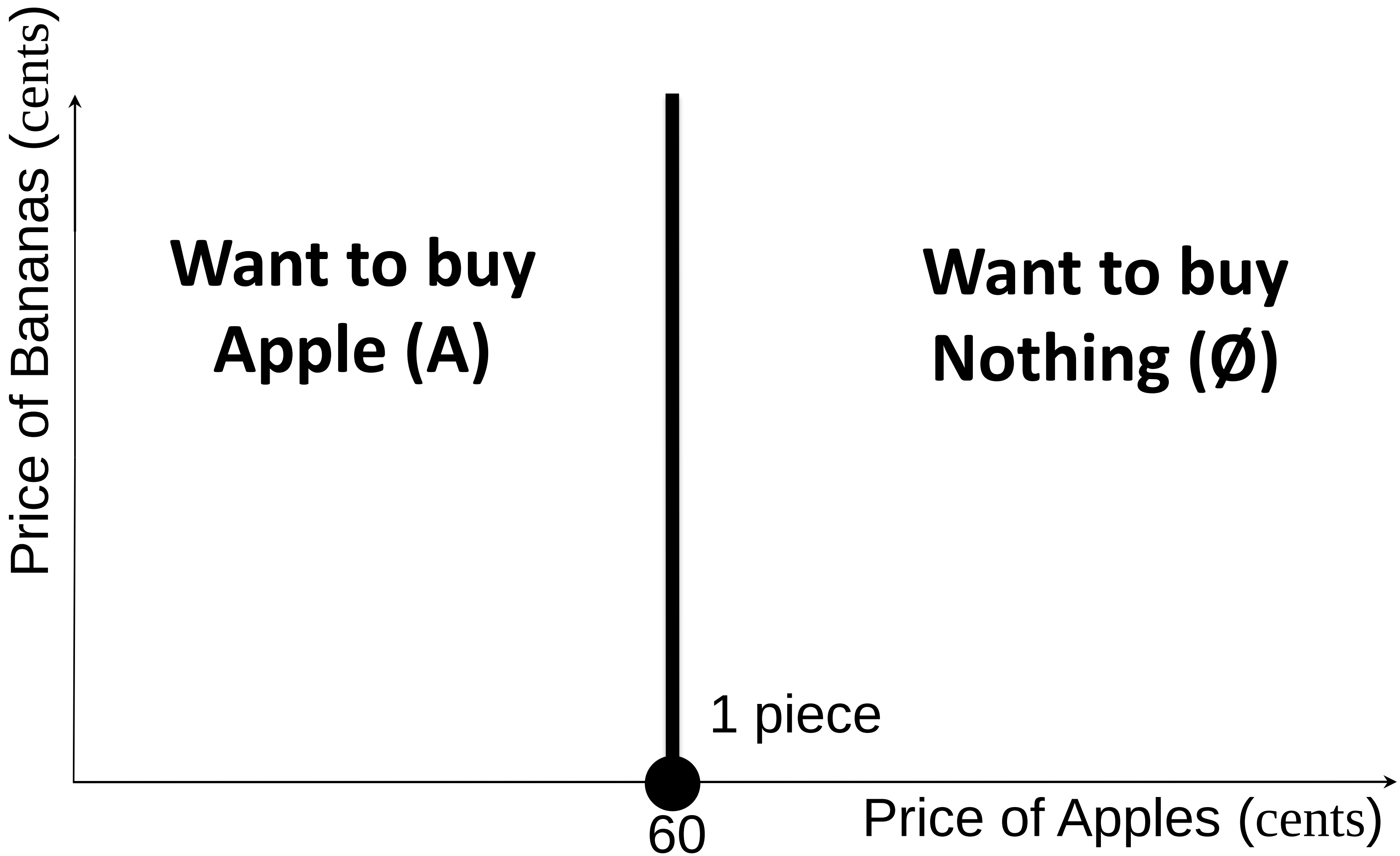
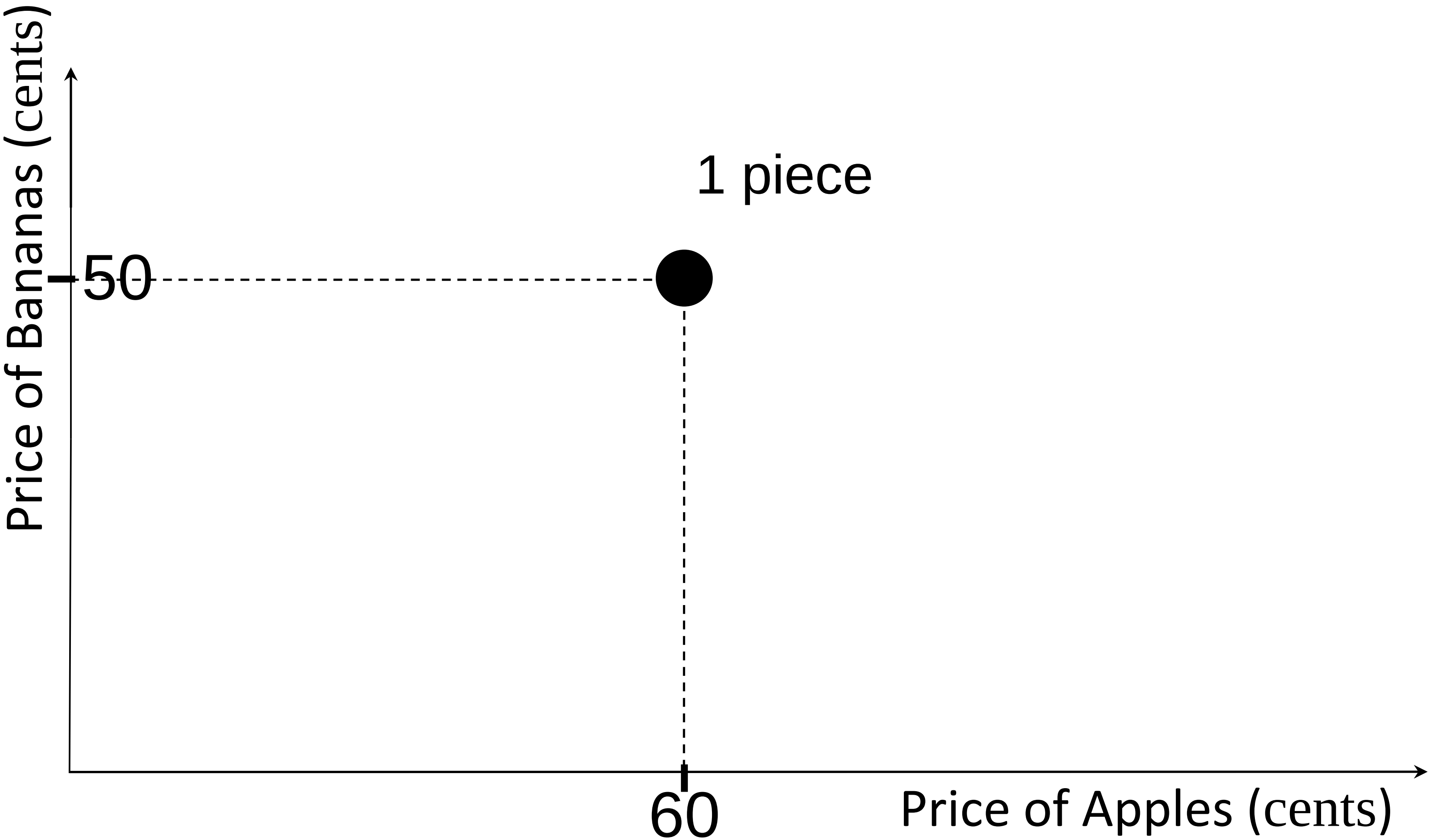


Example of a bid for (at most) a single piece of fruit: this bid says “willing to pay up to 60¢ for an Apple, or up to 50¢ for a Banana”



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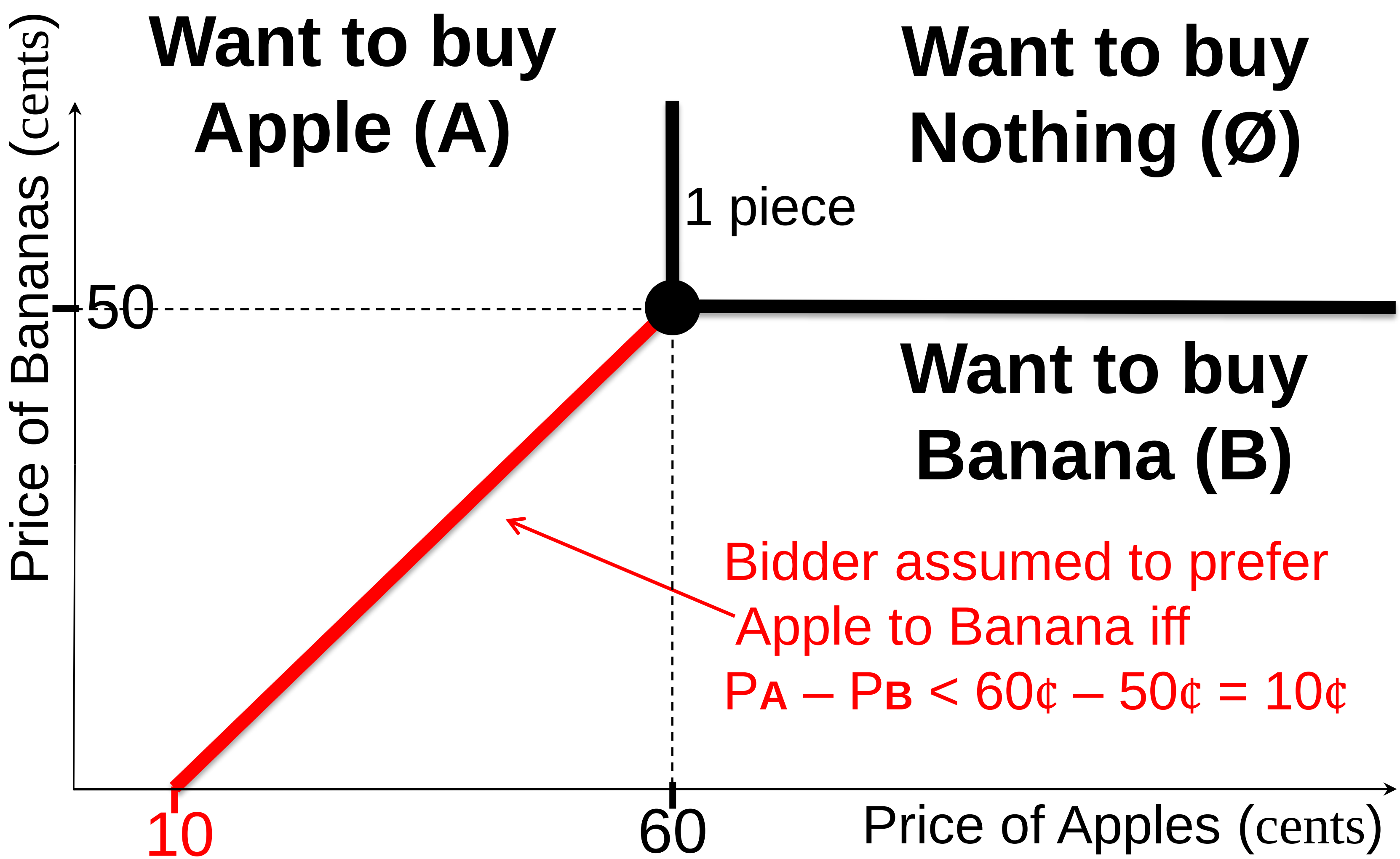
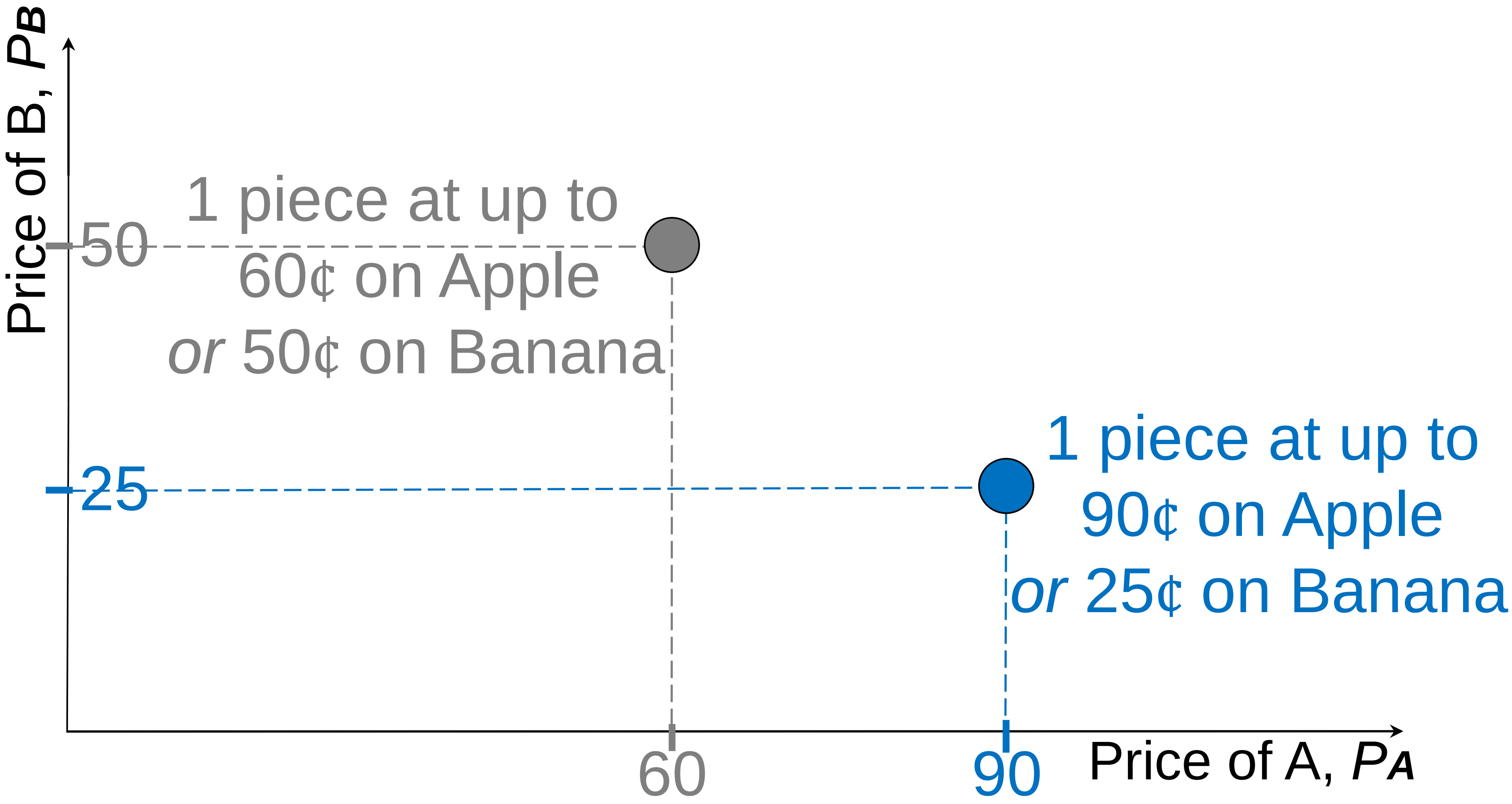
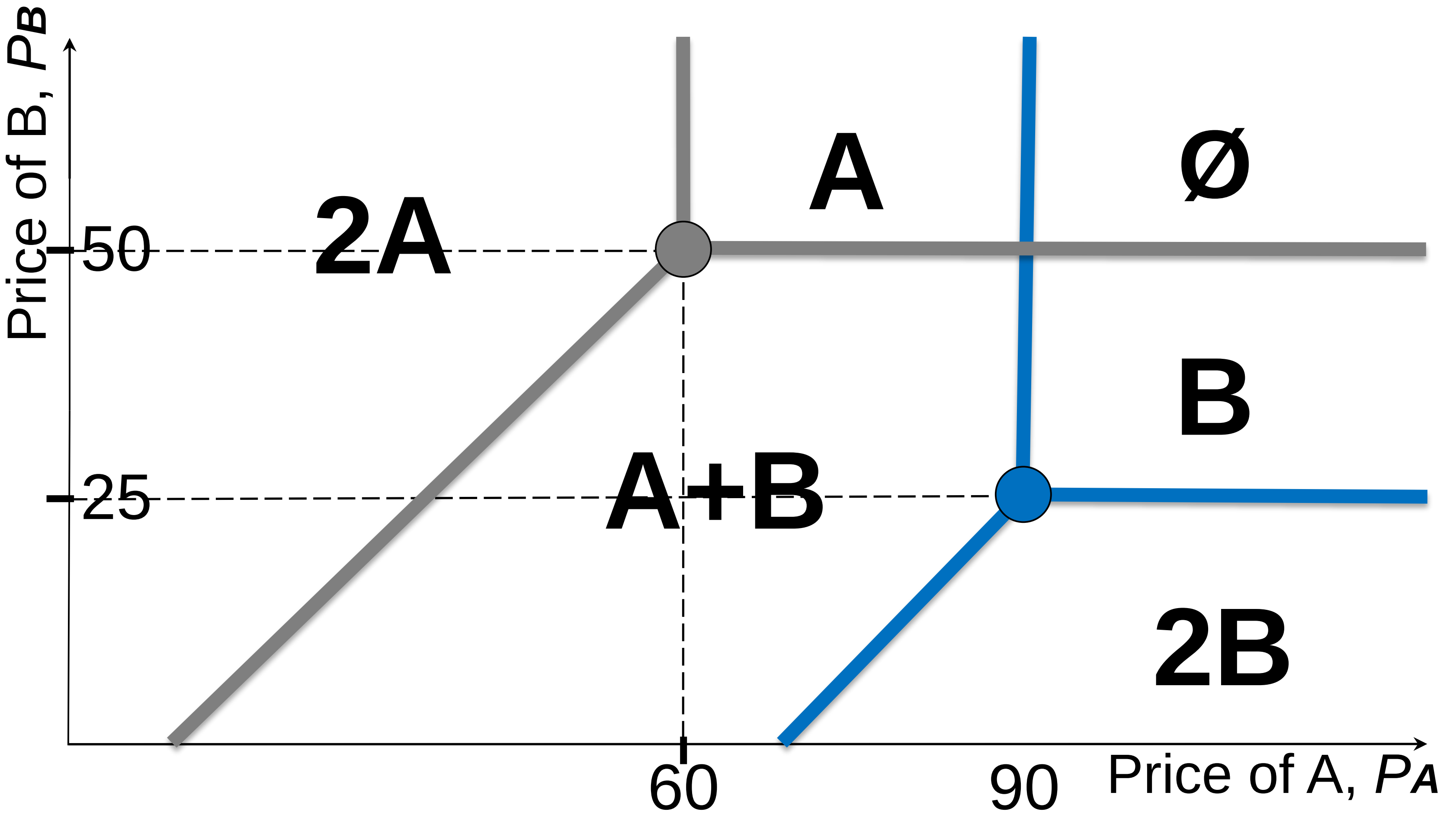


Example of bids for (up to) ***two*** pieces of fruit:
(the second is for my wife, who has an even stronger relative preference for apples)



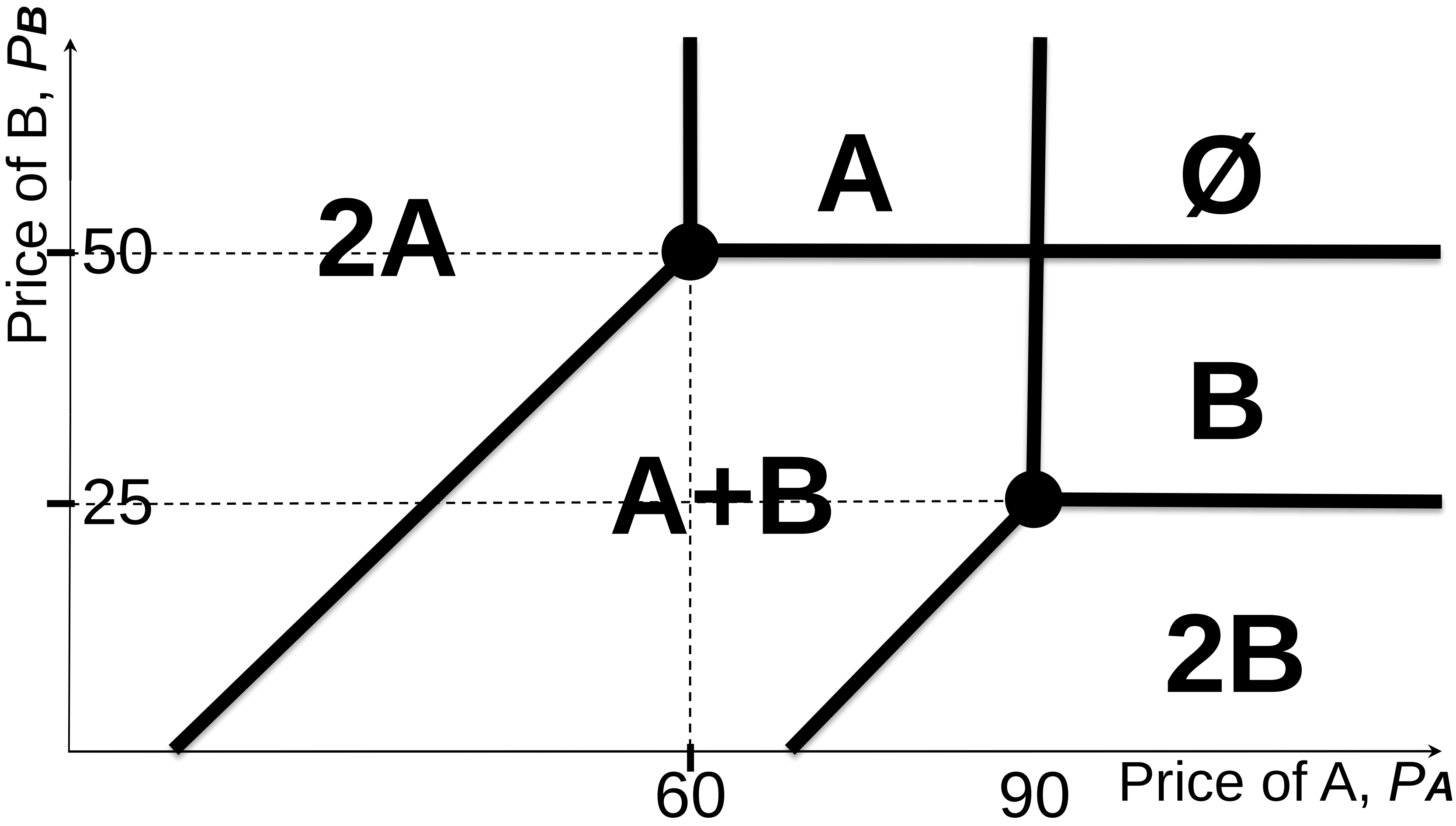
Example of bids for (up to) ***two*** pieces of fruit:
These bids will now results in these purchases, as function of the prices set by the auction



Example of bids for (up to) ***two*** pieces of fruit:

-- each bidder can give a list of ***MANY*** bids

-- note that the auction will treat two bids made by *different* bidders exactly like two bids made by the *same* bidder



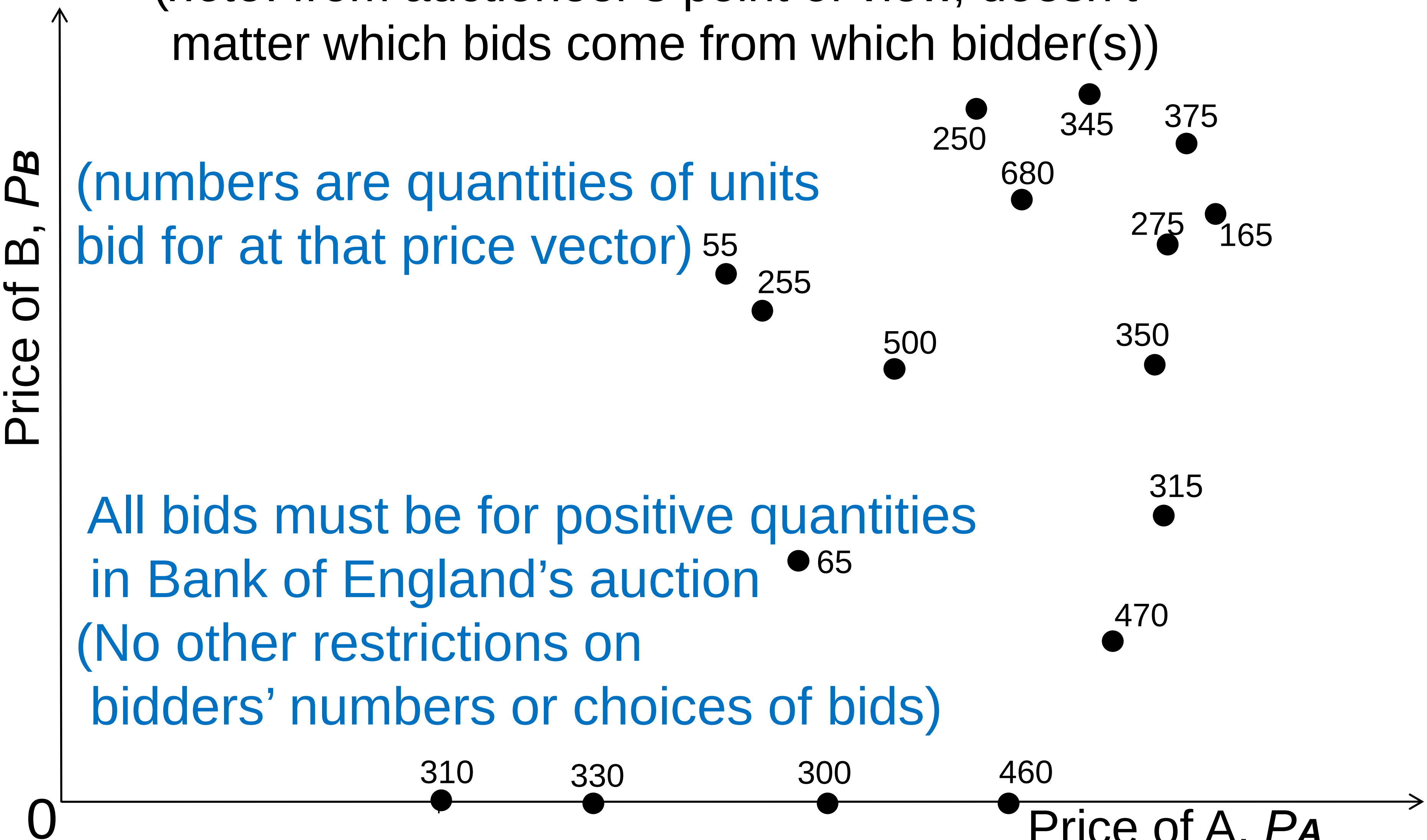
Bank of England's (original) bidding language

A bid is a list (of any length) of price vectors, and an associated quantity for each price vector, e.g., (60,50; 1), (90,25; 3), ….

Bank of England's (original) bidding language

Example of ***all*** the bids of ***all*** the bidders

(*note*: from auctioneer's point of view, doesn't matter which bids come from which bidder(s))



(numbers are quantities of units bid for at that price vector)

All bids must be for positive quantities in Bank of England's auction

(No other restrictions on bidders' numbers or choices of bids)

Bank of England's (original) bidding language

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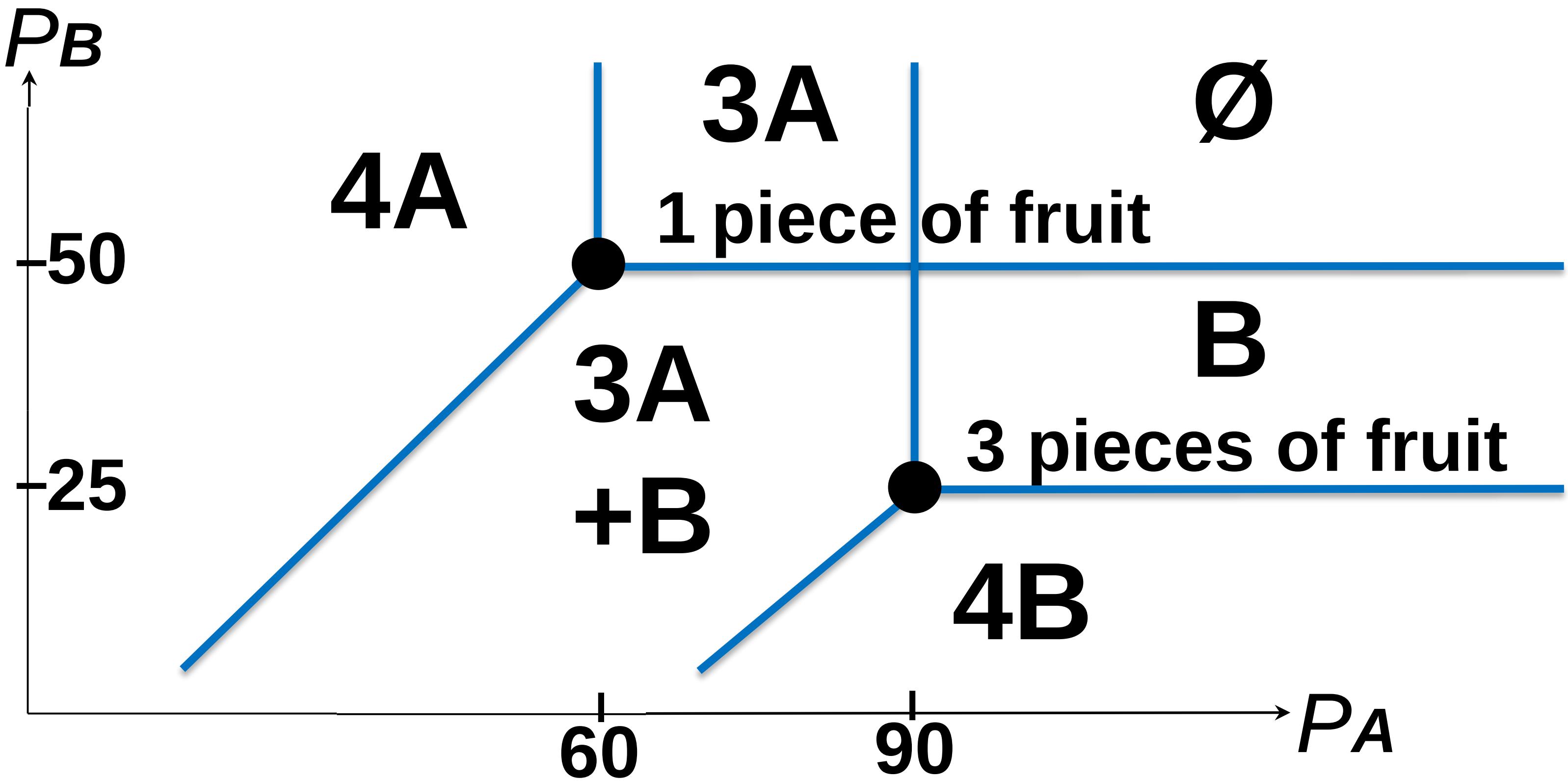
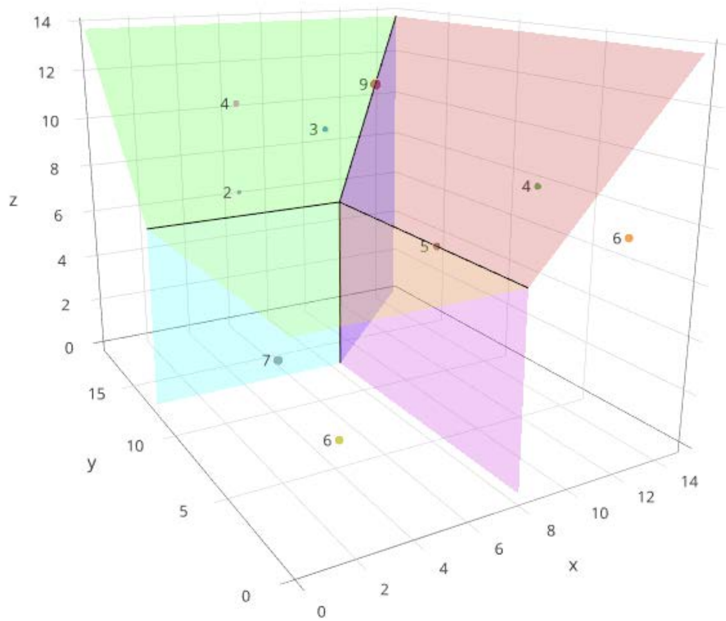


The BoE's *current* auctions sell more varieties which permits prettier pictures…



(3 varieties)

Note: the geometric objects we build are "Tropical Hypersurfaces"
so analyse using tropical geometry
--see Baldwin and Klemperer (2019)

& call this language "tropical"

Different languages for different contexts:

Iceland wanted (2015-16) to buy blocked "offshore" accounts in return for choices of ISK, or Euro bonds, or cash

each bidder (account-owner) has a **fixed budget** (i.e., fixed amount of money in her account)

→ I proposed "**Arctic** Product-Mix Auction" (Klemperer, 2018)

Different languages for different contexts:

Iceland *vs* Bank of England

Choosing between alternatives in "OR" bids:

Both cases: "pay up to V_A for good A, **OR** up to V_B for good B"

BoE: bidder prefers good A if : $\mathbf{V_A - P_A > V_B - P_B}$ (& $P_A < V_A$)

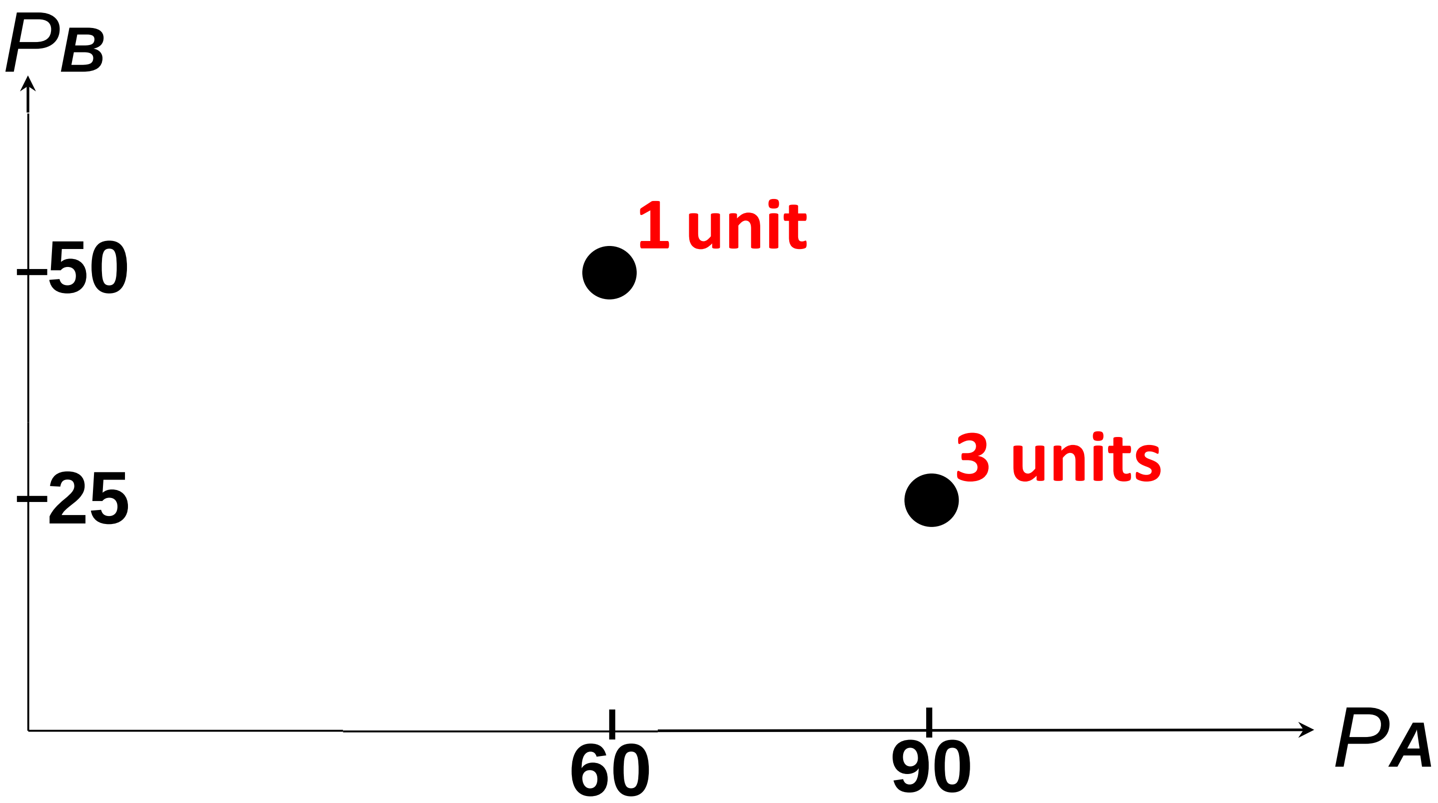
Iceland: bidder who spends budget B

prefers good A if $(V_A - P_A)(B/P_A) > (V_B - P_B)(B/P_B)$ (& $P_A < V_A$)

i.e., bidder prefers good A if : $\mathbf{V_A/P_A > V_B/P_B}$ (& $P_A < V_A$)

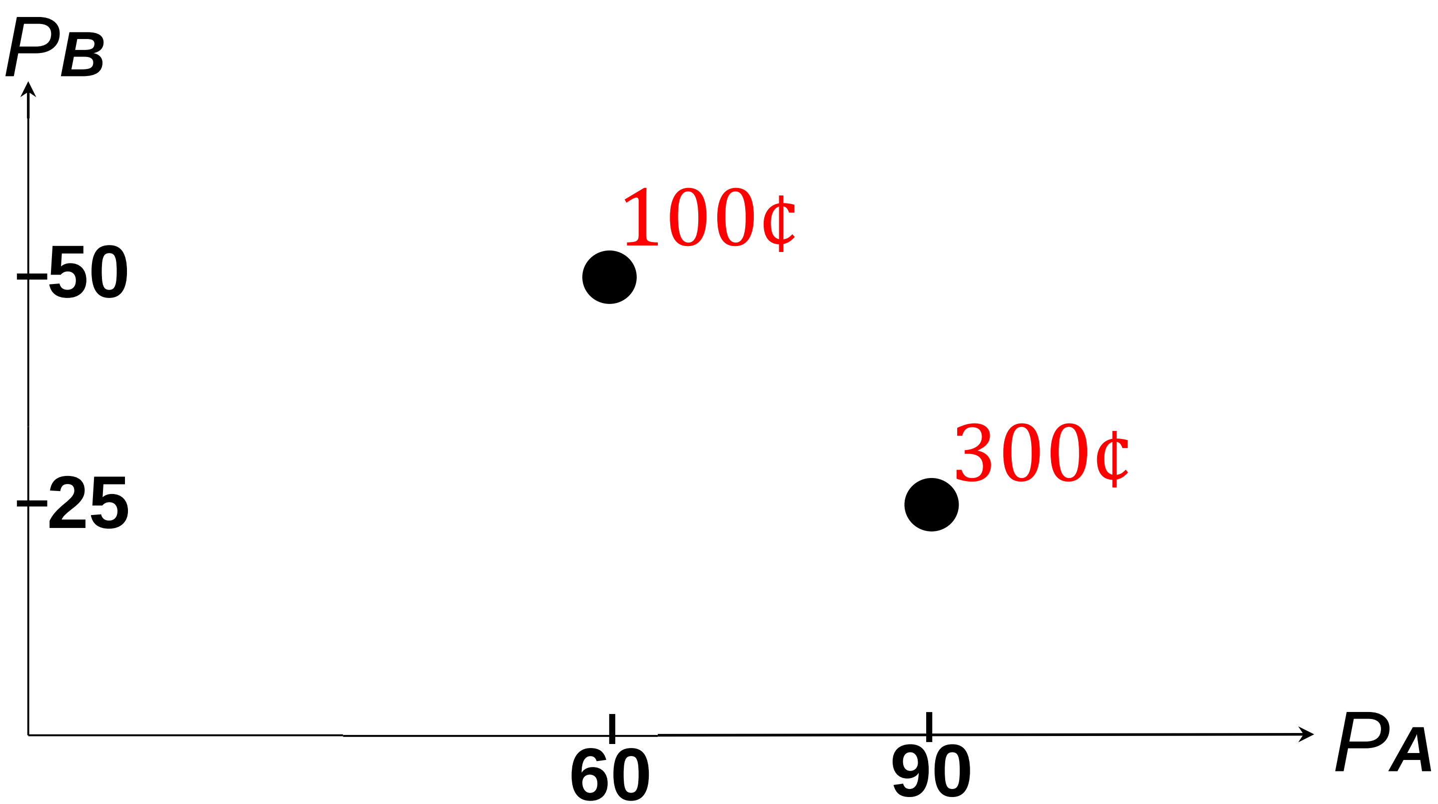
Bank of England's "tropical" language

A tropical bid is a list (of any length) of price vectors, and an associated quantity for each price vector, e.g., (60,50; 1 unit), (90,25; 3 units), ...



Iceland's "arctic" language (bidders have fixed budgets)

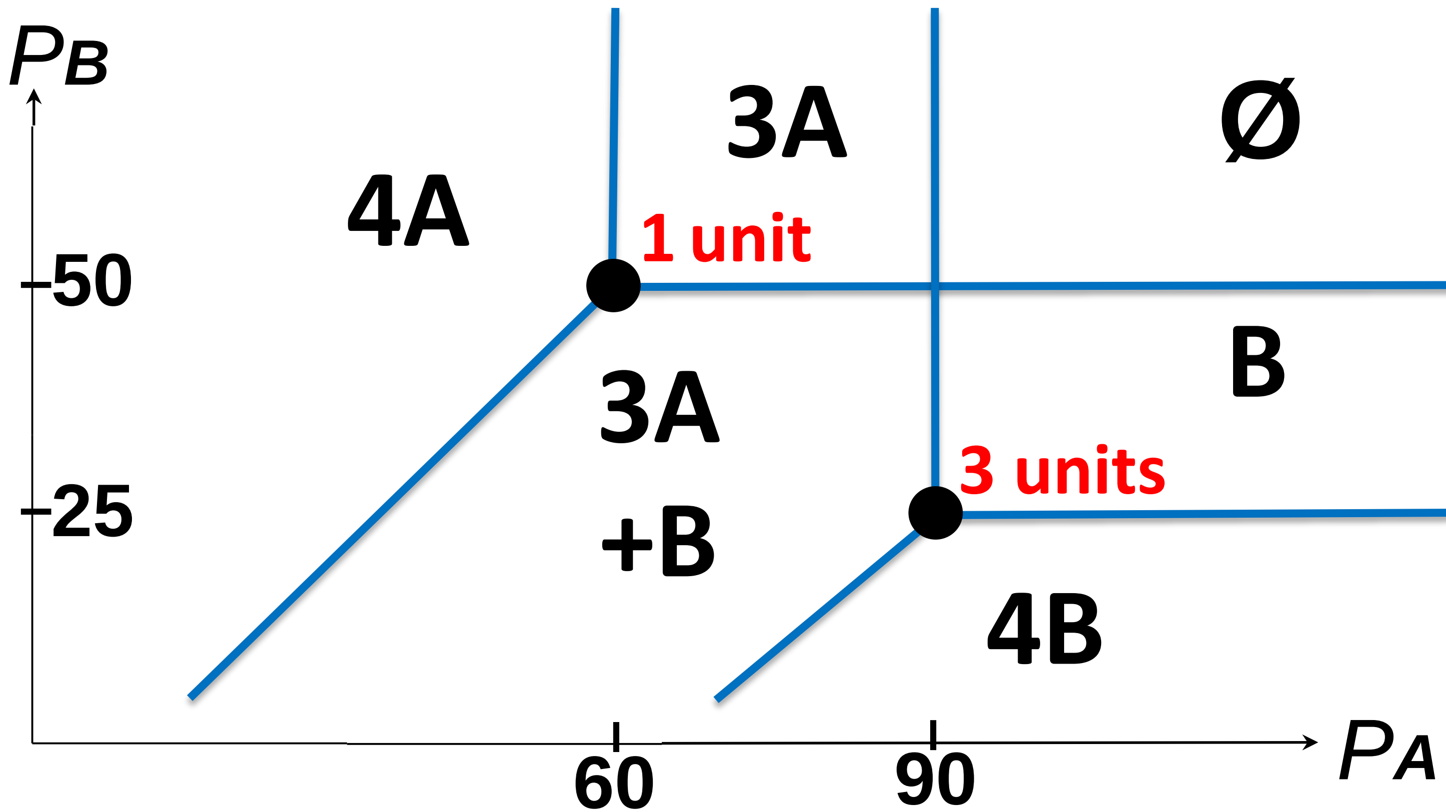
An arctic bid is a list (of any length) of price vectors, and an associated budget for each price vector, e.g., (60,50; 100¢), (90,25; 300¢), …



Bank of England's "tropical" language

[*Recall, a bid* (**V_A, V_B**; ·) *wants A if* **$V_A - P_A > V_B - P_B$** (& $P_A < V_A$)]

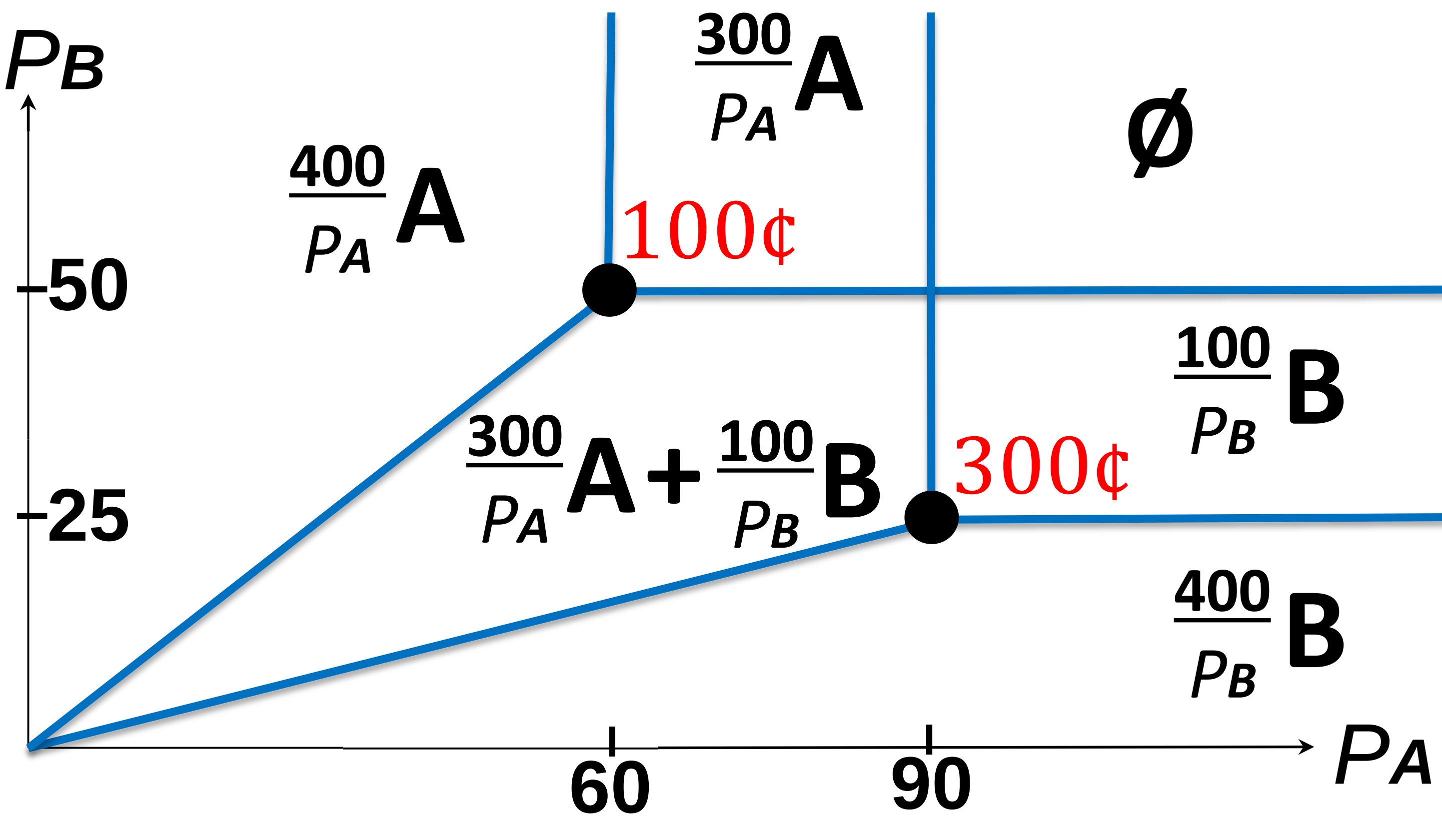
Purchases resulting from (60,50; 1unit), (90,25; 3units):



Iceland's "arctic" language (bidders have fixed budgets)

[*Recall, a bid* ($V_A, V_B; \cdot$) *wants A if* $V_A/P_A > V_B/P_B$ (& $P_A<V_A$)
vs., for Bank of England wants A if $V_A - P_A > V_B - P_B$ (& $P_A<V_A$)]

Purchases resulting from (60,50; 100¢), (90,25; 300¢):



Emerging-Market Debt Crisis

IMF staff suggesting variant of Arctic PMA
may help debt restructuring.

Give creditors' "budgets" proportional to their current claims;
they bid their budgets for different alternative claims:

--early vs. late repayment

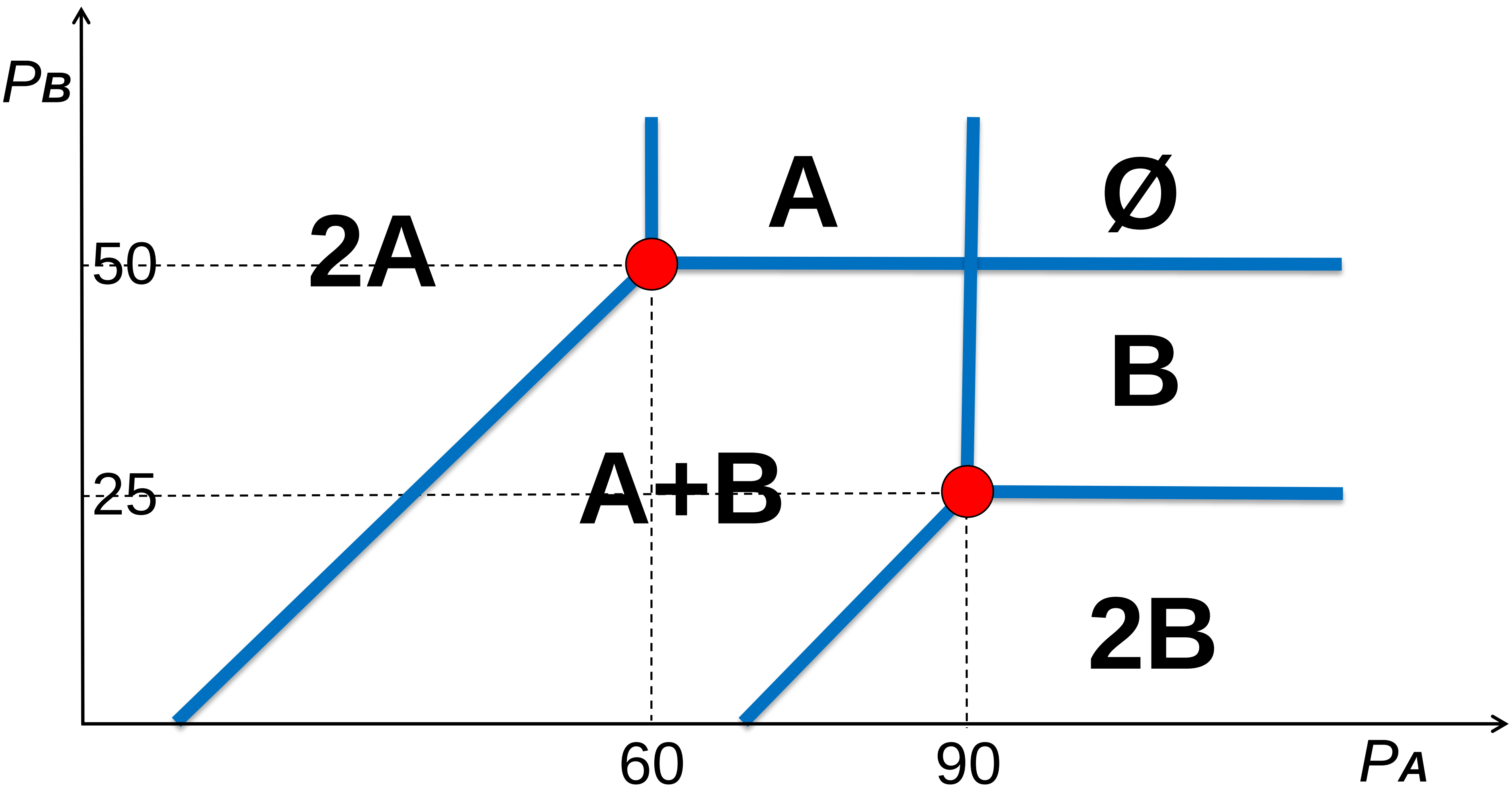
--repayment in $ vs. local currency

--fixed repayment vs. GDP linked

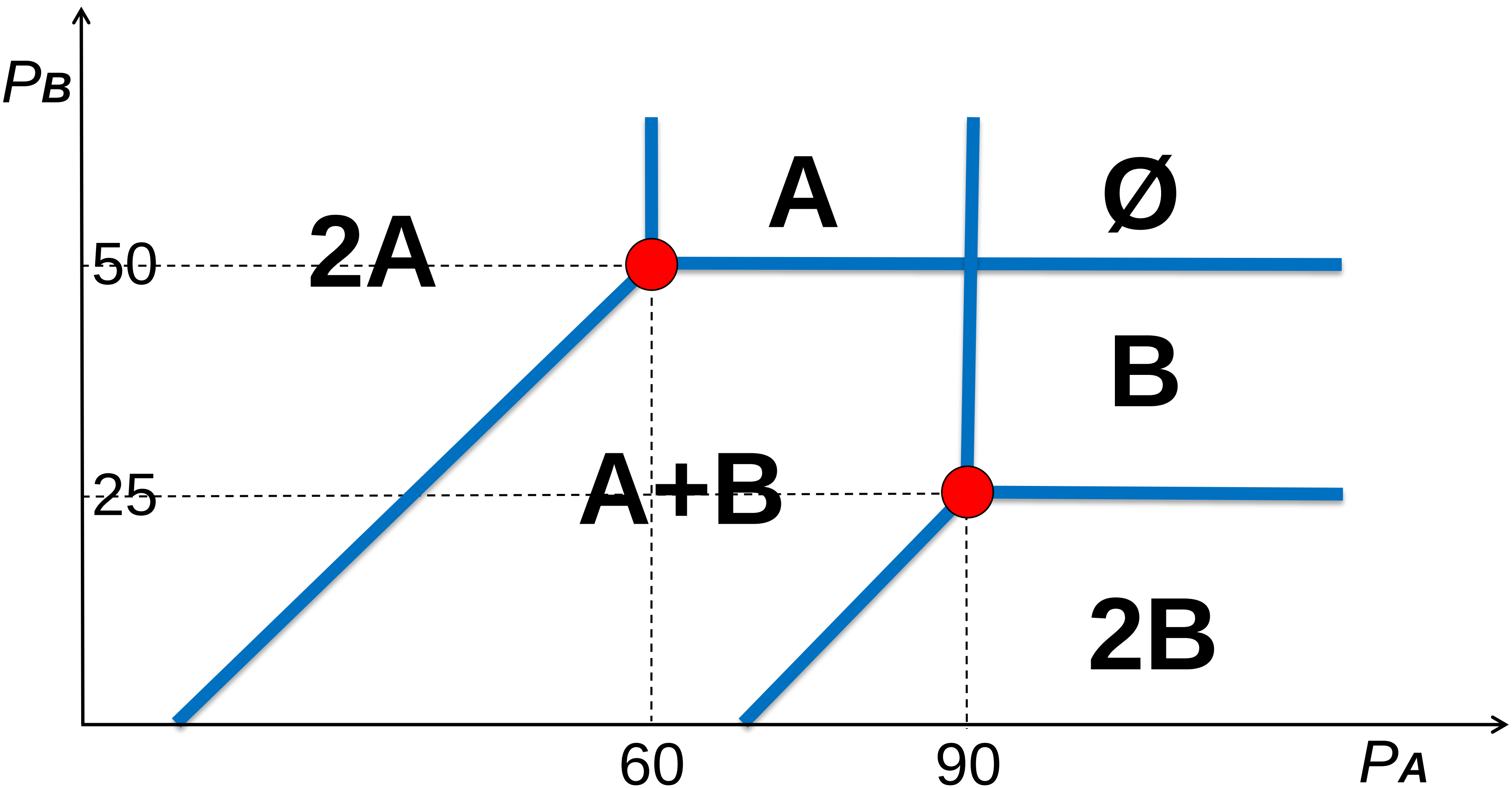
--etc.

Different languages for different contexts:

e.g. developing Bank of England's tropical language



Can the bidder express ***any*** quasilinear preferences with 1:1 trade-offs between goods, using only tropical bids?



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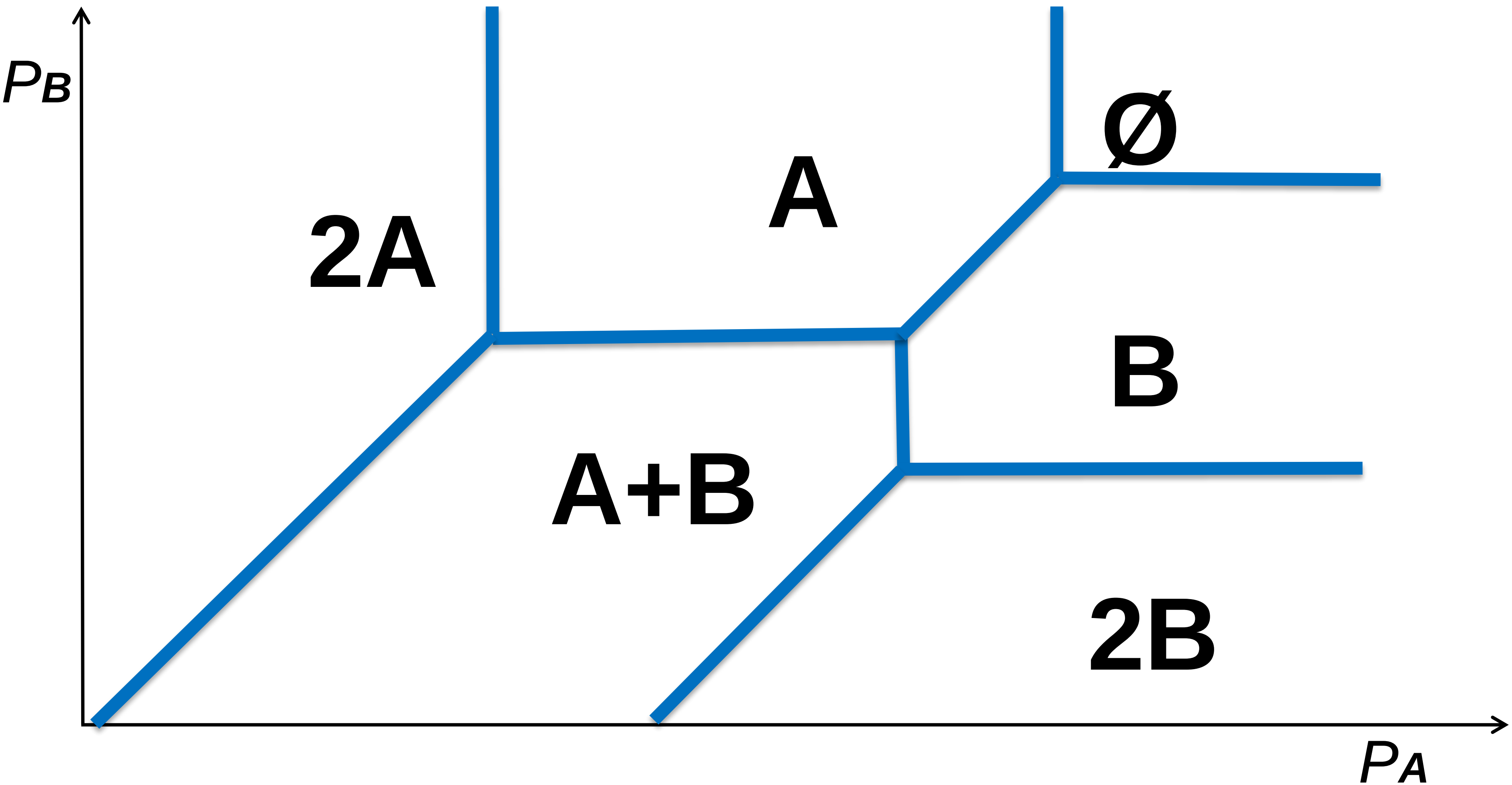
Problem—can we express these preferences?



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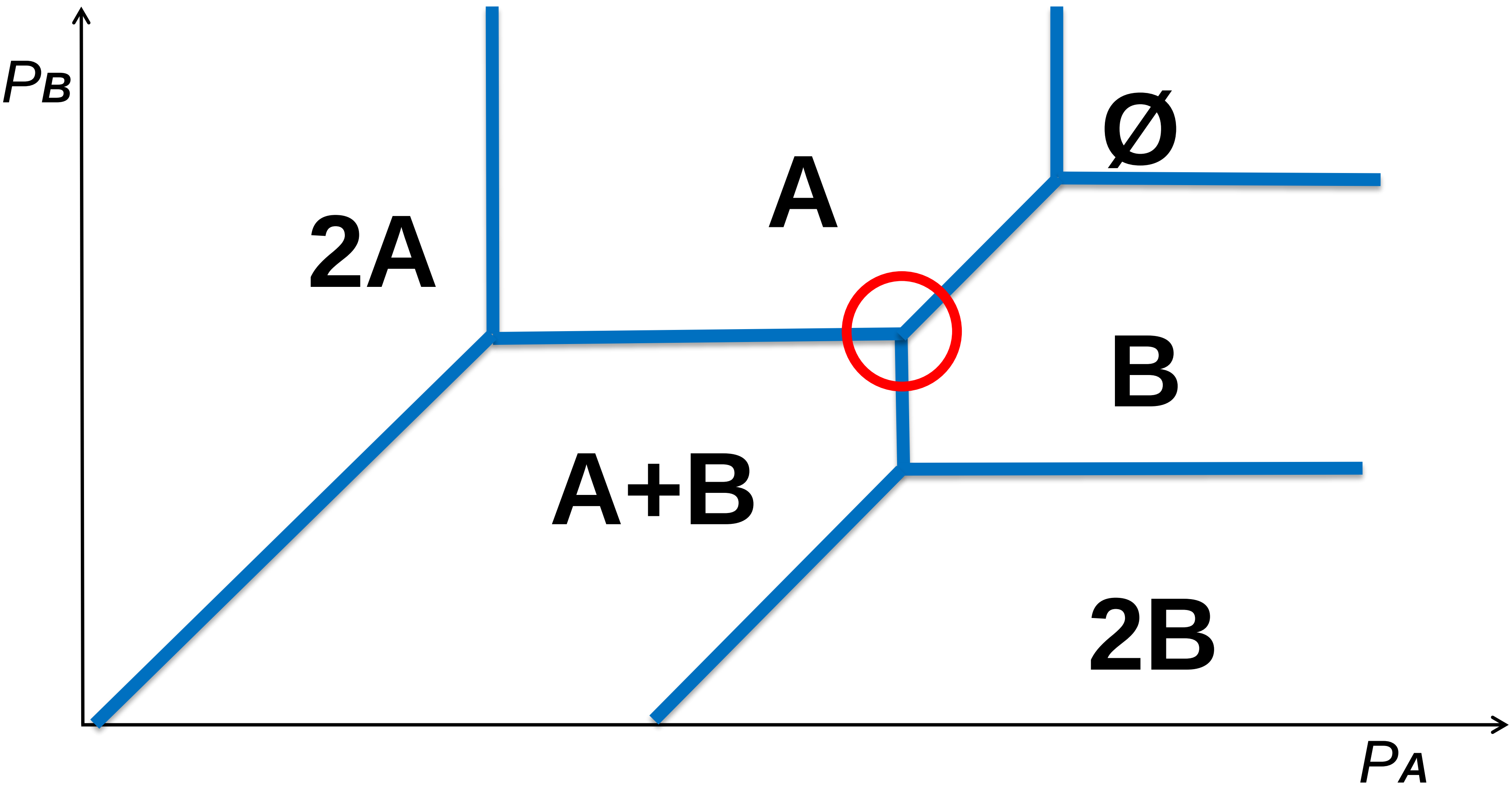
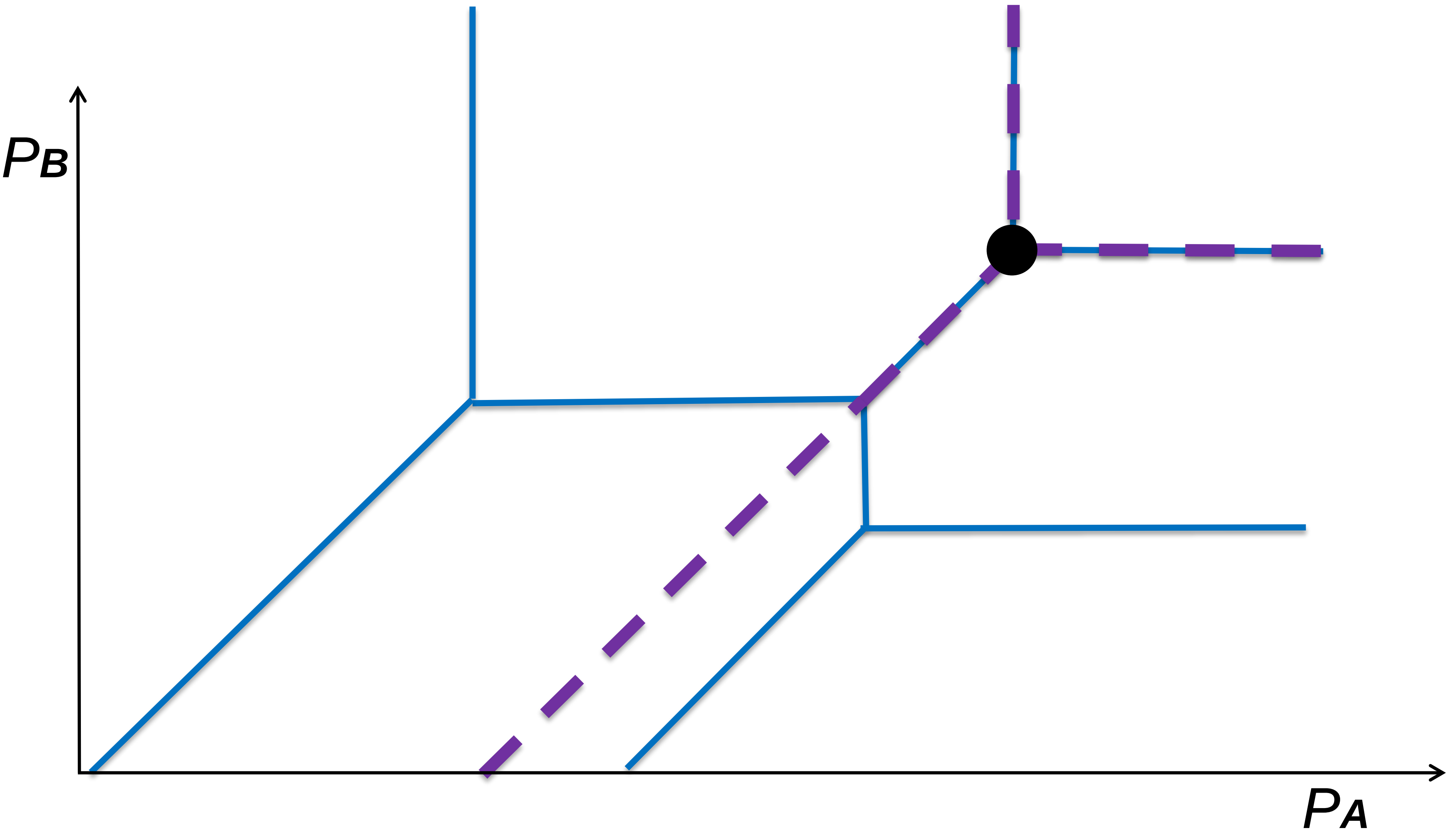
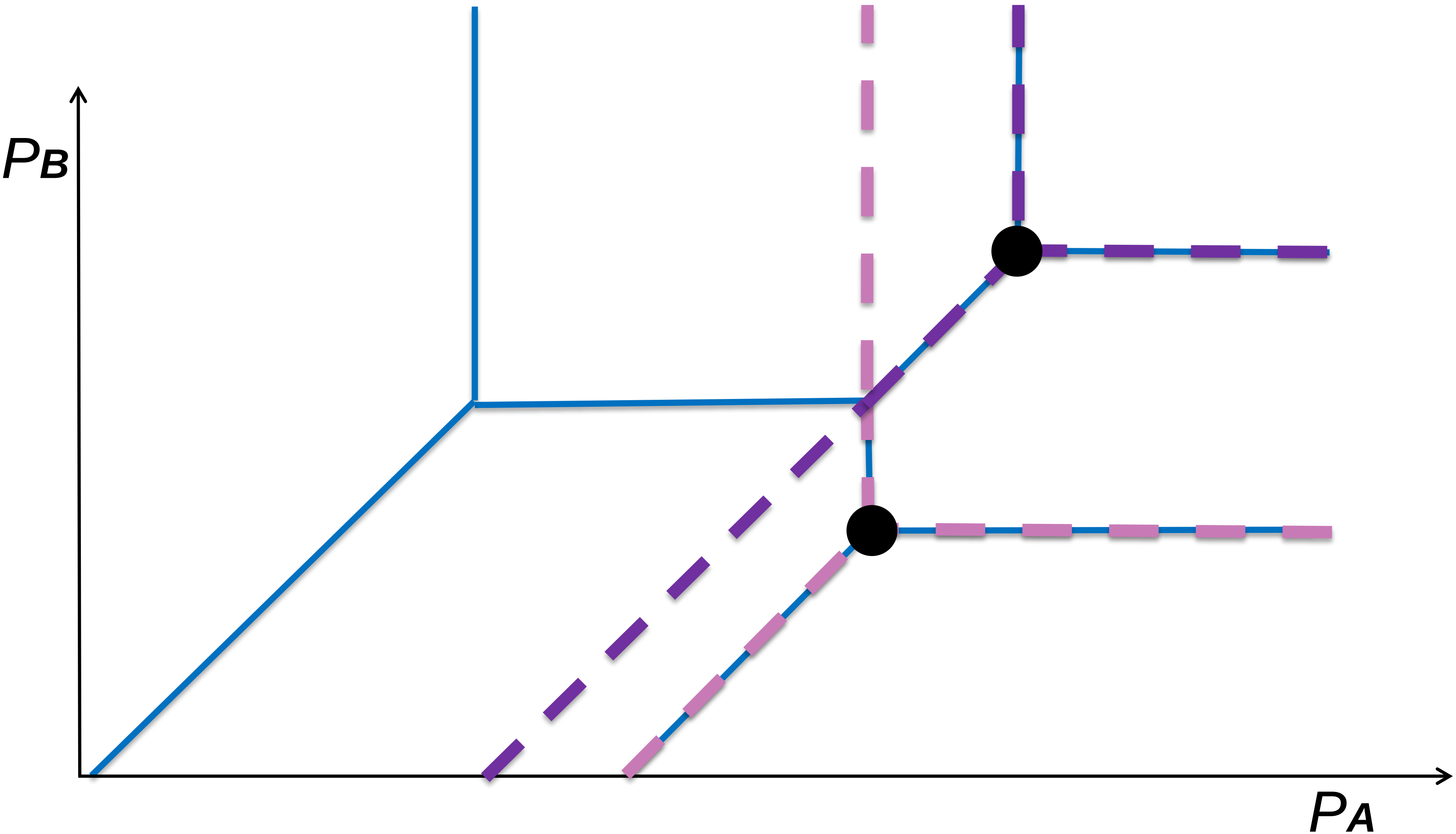


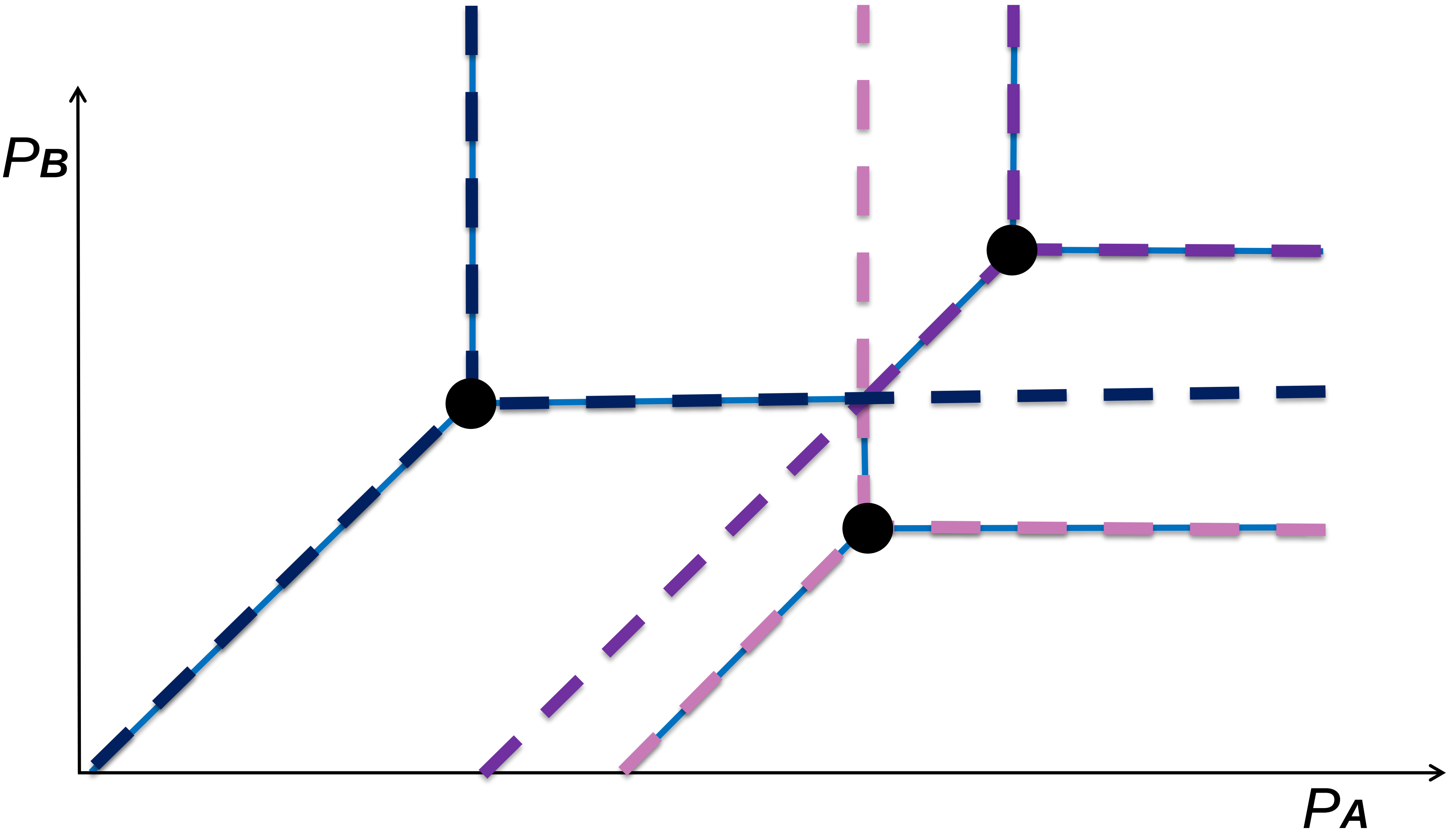


P_B
P_A

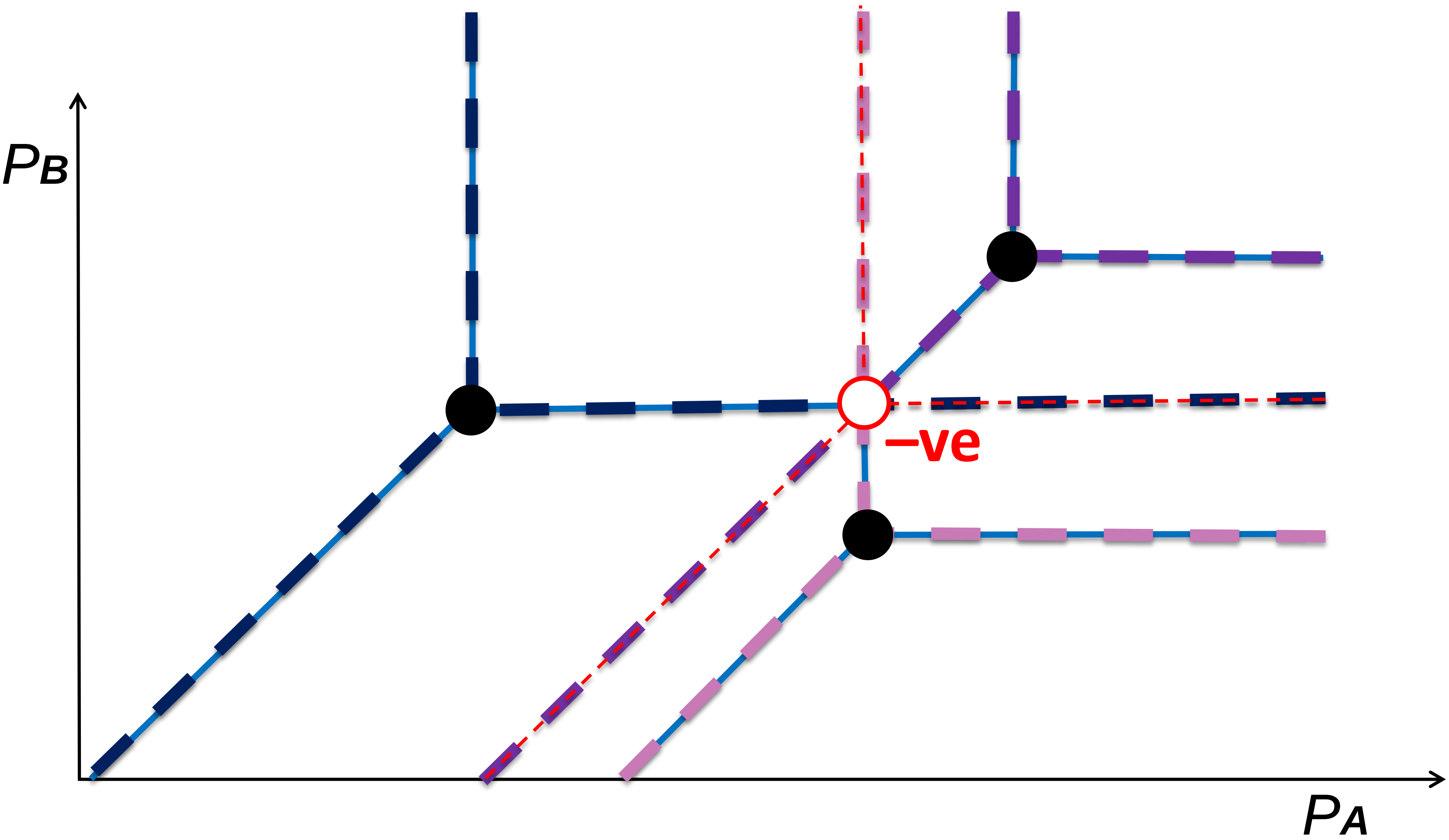


PB
PA

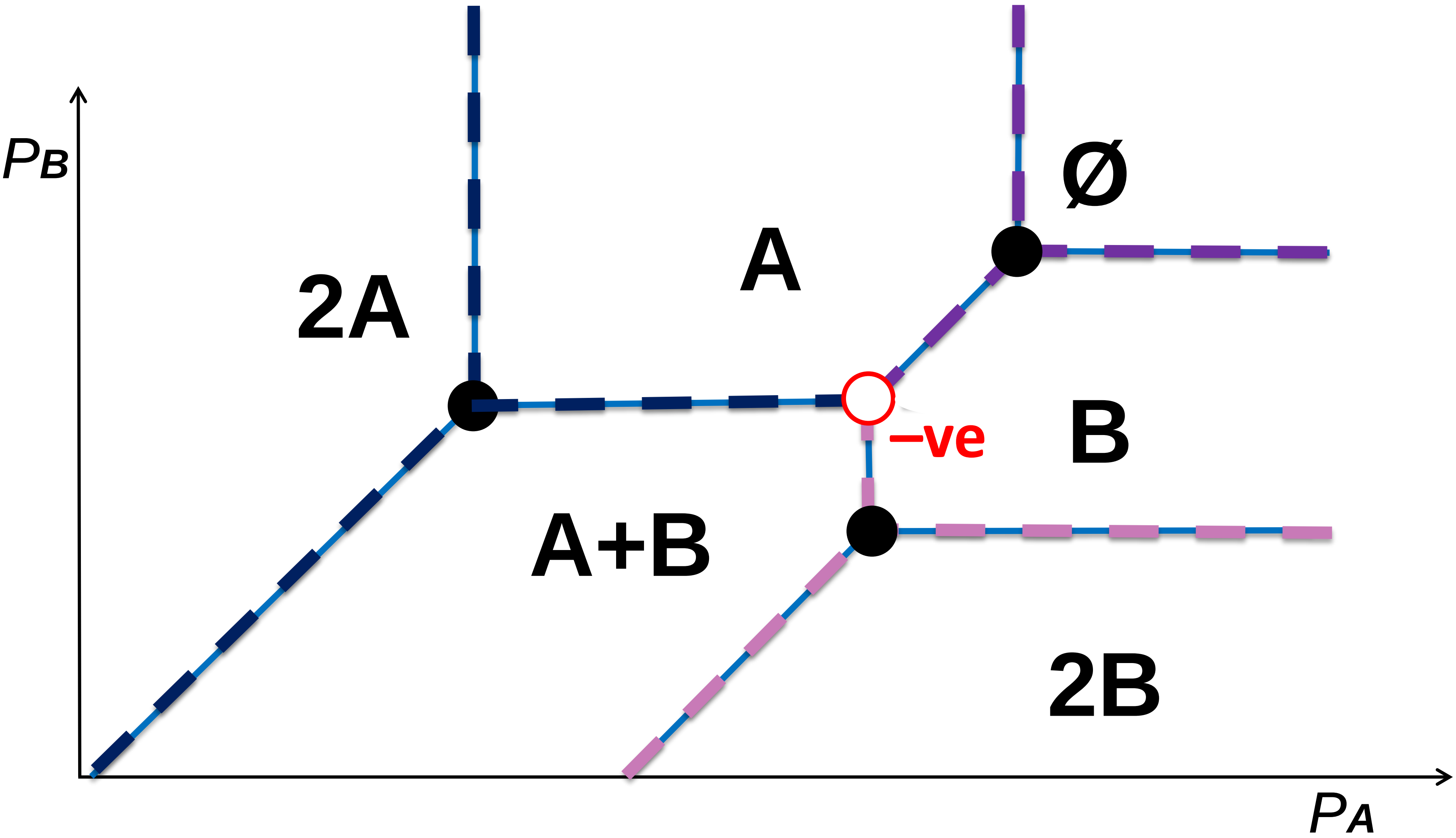
These three tropical bids gives us too much …



These three tropical bids gives us too much …
but subtracting a bid cancels what we don't need

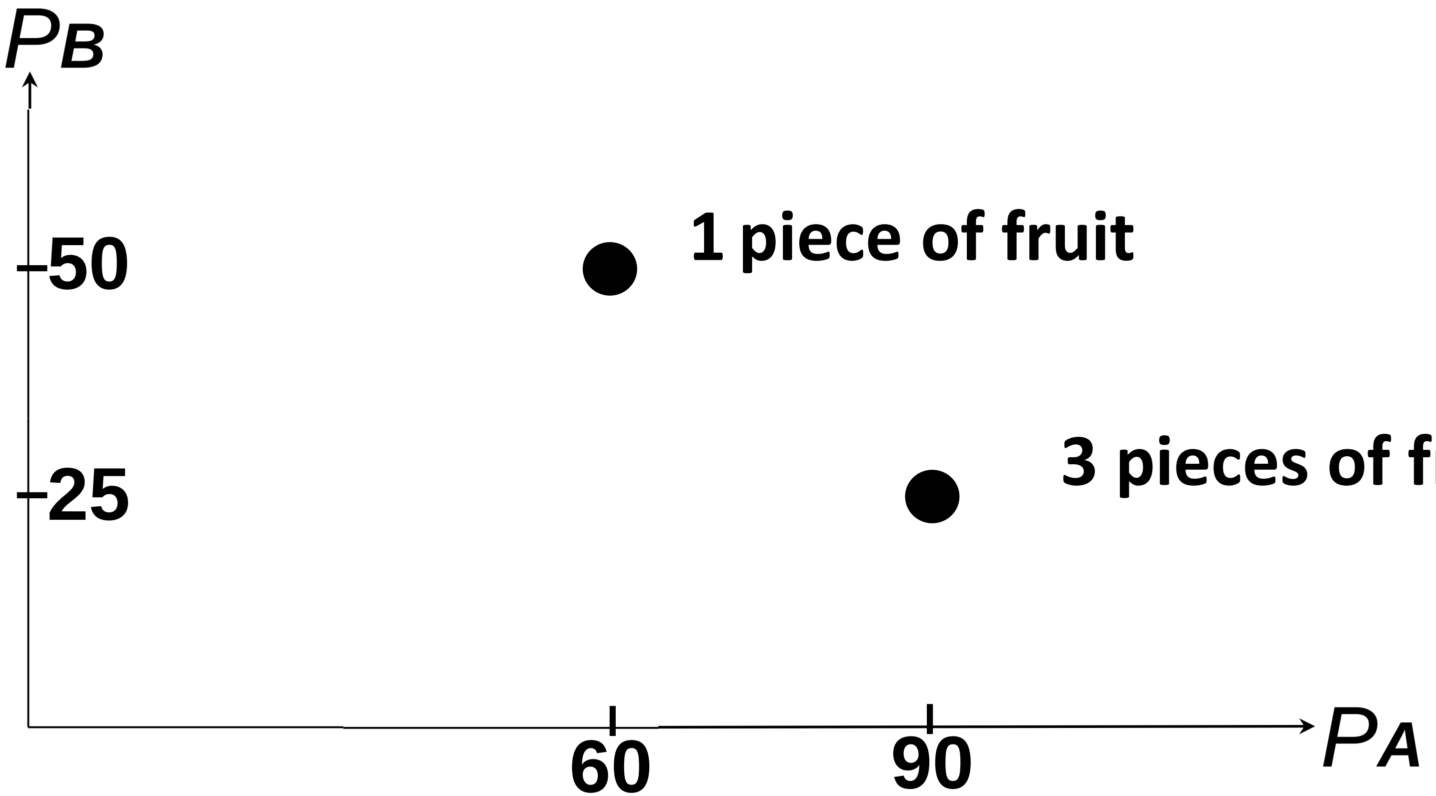


→ These three tropical bids ***minus*** a bid…
gives the preferences we wanted to represent!

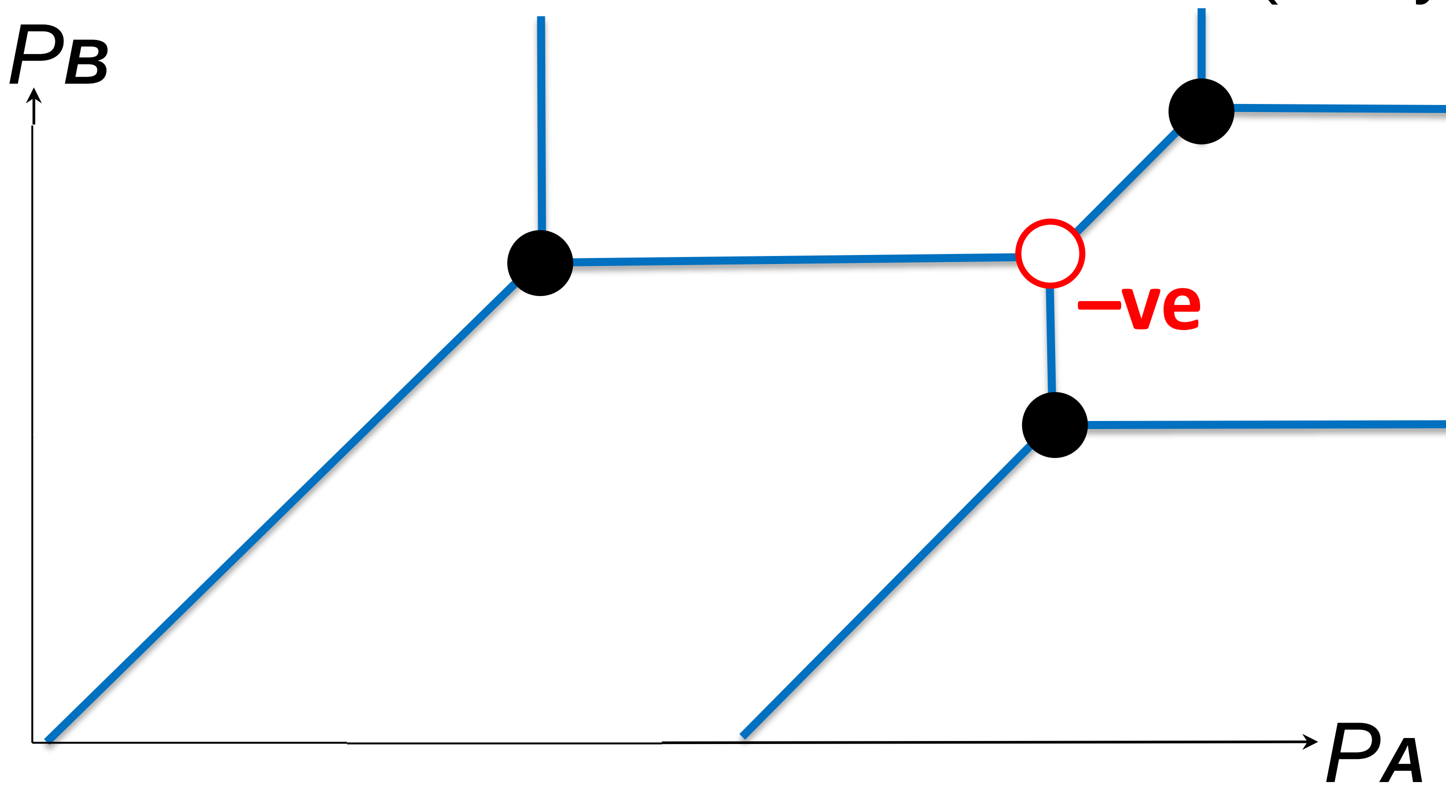


Generalising Bank of England's language

A (basic) tropical bid is a list (of any length) of price vectors, and an associated quantity (maybe –ve) for each price vector, e.g., (60,50; 1), (90,25; 3), ….

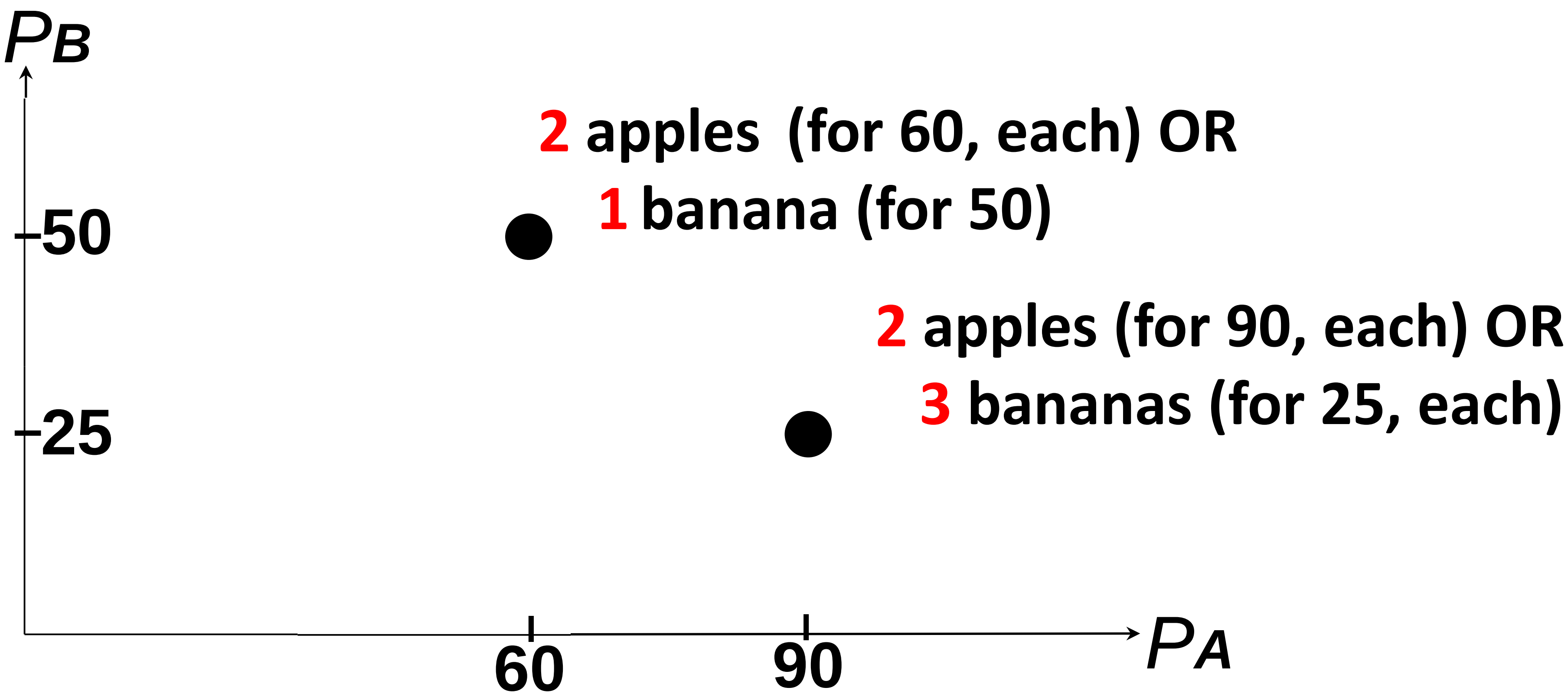


Any bidder with 1-1 trade-offs everywhere
(= “strong substitutes preferences”,
see Milgrom and Strulovici, 2009)
can *perfectly* represent its preferences using
(basic) positive and negative tropical bids.
No other proposed “language” for these preferences does this
(to my knowledge)



Further generalising Bank of England's language

A *generalised* tropical bid is a list (of any length) of price vectors, and an associated **quantity vector** for each price vector, e.g., (60,50; **2,1**), (90,25; **2,3**), ….



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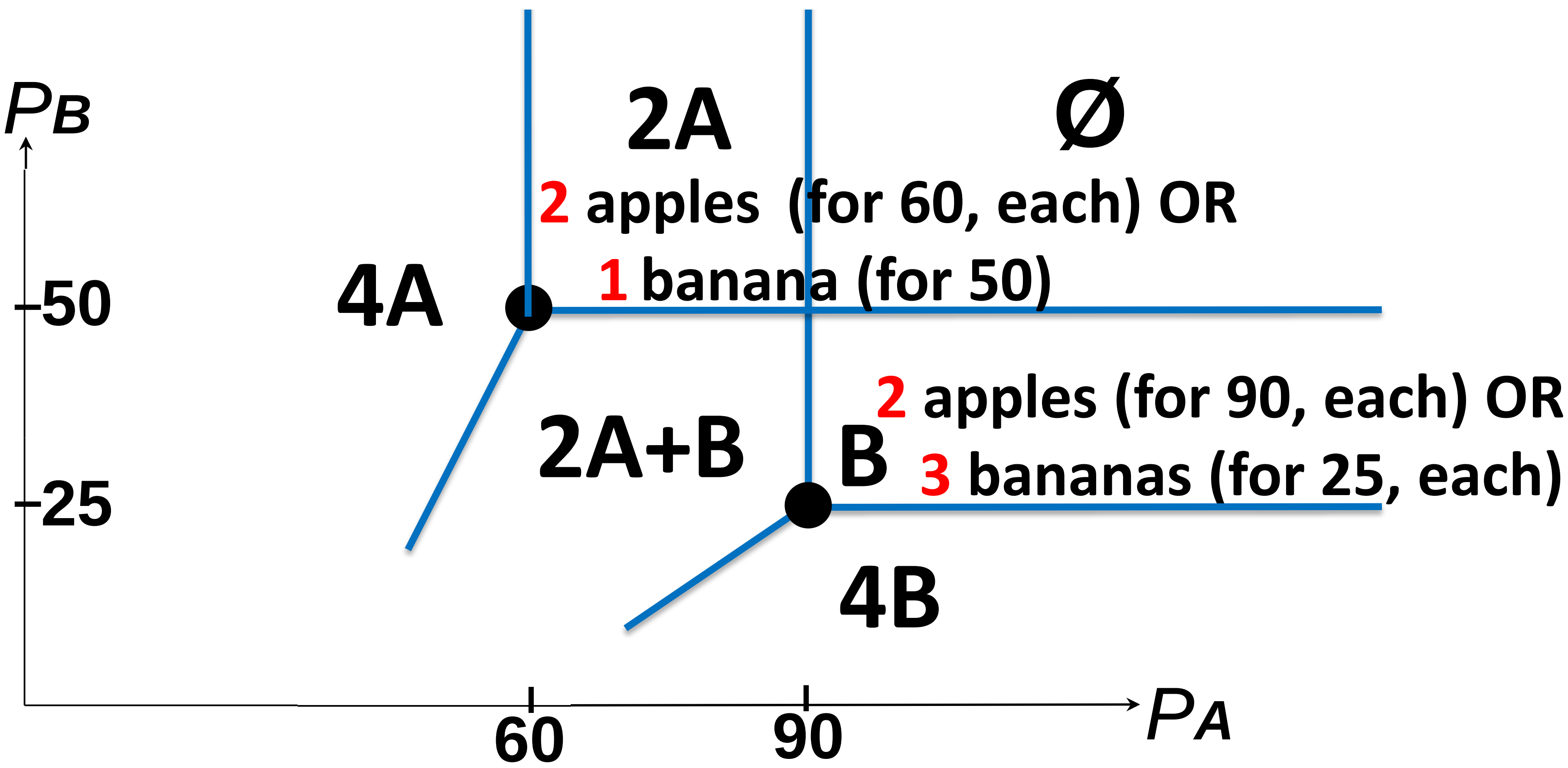
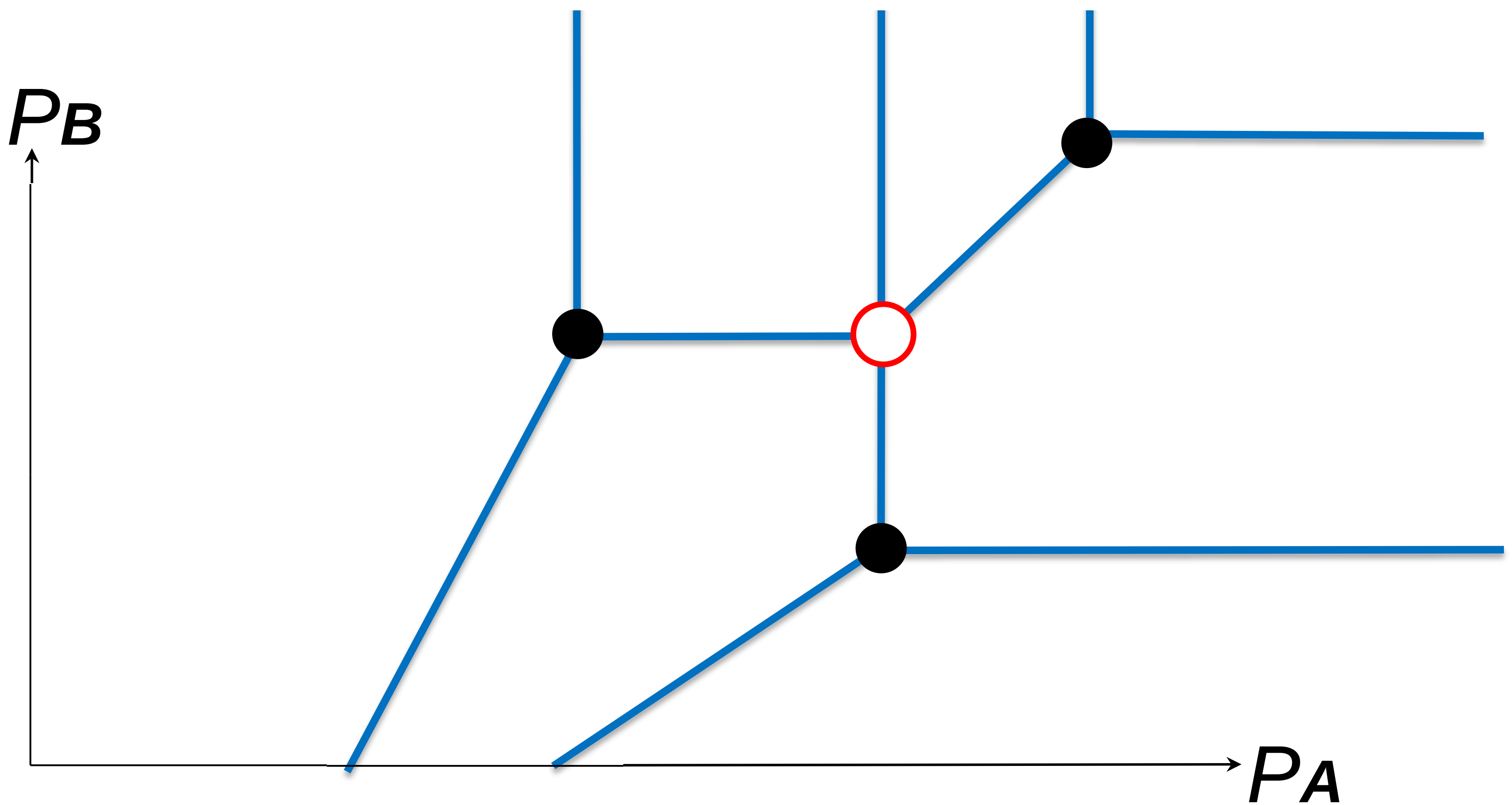


Any bidder with ~~strong~~ ***any*** substitutes preferences can *perfectly* represent its preferences using ~~(basic)~~ ***generalised*** positive and negative tropical bids.



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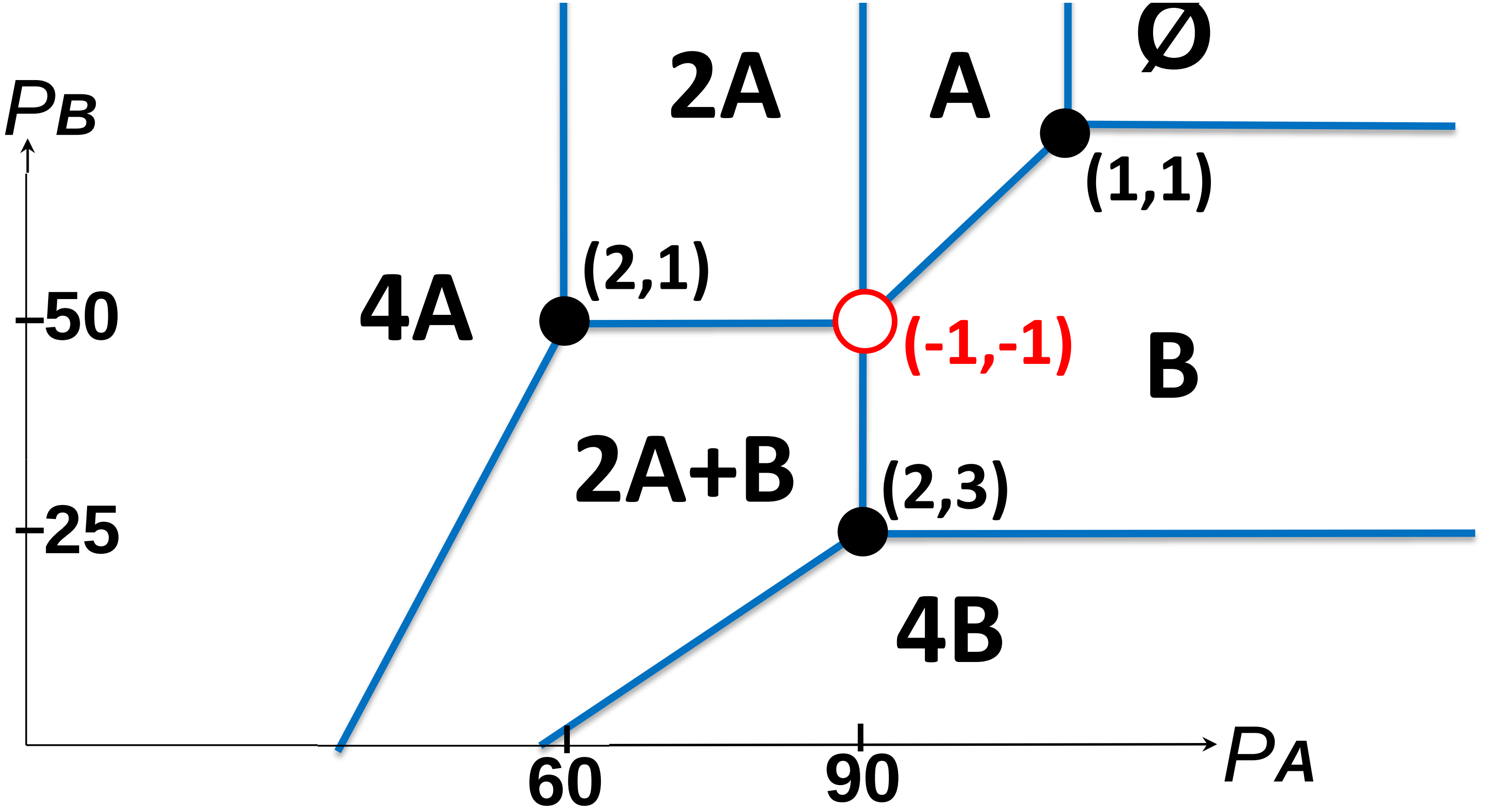
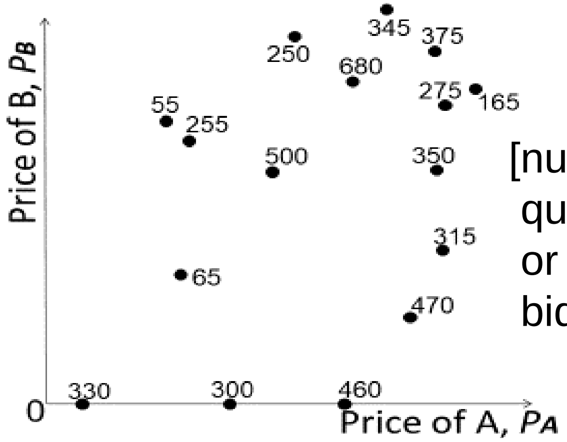


Any bidder with ~~strong~~ ***any*** substitutes preferences can *perfectly* represent its preferences using ~~(basic)~~ ***generalised*** positive and negative tropical bids.

Most-recent update of Bank of England's auction allows some other forms of preferences, permitting total supply to depend on bidding.

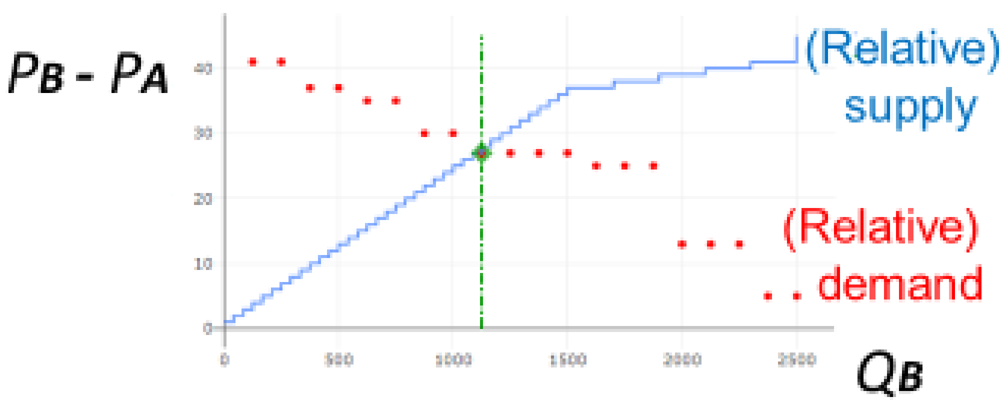
Given bids, how find prices & allocations?



[numbers are:
quantities of units (if tropical),
or budgets (if arctic),
bid for at the price vector)]

For small number of goods (2 or 3), graphical solutions elucidate relationship to competitive equilibrium

--see Klemperer (2008, 2010) or film on my website



"A marvellous application of theoretical economics to a practical problem of vital importance"

-- Mervyn King, then-Governor,
(quoted in *the Economist*)

"A major step forward in practical policies to support financial stability"

-- Paul Fisher, Executive Director

Mark Carney, next Governor

-- expanded the role of the auctions (in 2013)

-- introduced updated version with more options for auctioneer (in 2014*)

*this version implemented with Elizabeth Baldwin

Given bids, how find prices & allocations?

For Bank of England auction with only +ve bids

Competitive equilibrium maximises welfare:

R_j of good j to allocate, $j = 1, \ldots, \mathrm{N}$;

bid i is for k_i units at (up to) price $(p_{i1}, \ldots, p_{iN})$, $k_i > 0$

So allocate $x_{ij} \geq 0$ of good j to bid i

$$\text{to } \max \sum_i \sum_j p_{ij} x_{ij}$$

$$\text{s.t. } \sum_j x_{ij} \leq k_i \quad \forall\, i \quad \text{[bid constraints]}$$

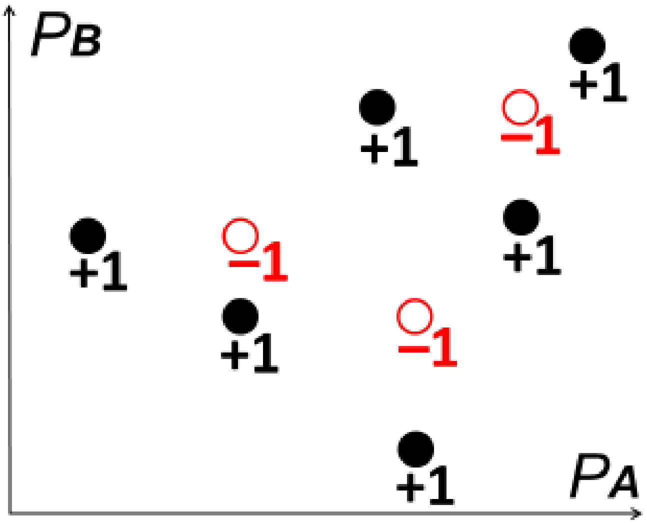
$$\text{and } \sum_i x_{ij} \leq R_j \quad \forall\, j \text{ [resource constraints]}$$

Linear program gives us all prices and allocations
(good *j*'s price is shadow price on *j*th resource constraint)

Given bids, how find prices & allocations?

Extending Bank of England solution to -ve bids
(paper with Baldwin, Bichler, Fichtl)

We still want to allocate $\underline{R}$ in total.



Let V($\underline{R}$+$\underline{S}$) be value of solution to LP for allocating $\underline{R}$+$\underline{S}$ to (only) +ve bids.

Let V*($\underline{S}$) be value of solution to LP for allocating $\underline{S}$ to (only) −ve bids.

Solution to whole program is V($\underline{R}$+$\underline{S}$) -V*($\underline{S}$) for correct $\underline{S}$.

But correct $\underline{S}$ must give same price vector for both programs, and price vector is marginal value of extra resources, $\underline{S}$.

i.e., we want a stationary point of V($\underline{R}$+$\underline{S}$) -V*($\underline{S}$)

→ solve min {V($\underline{R}$+$\underline{S}$) -V*($\underline{S}$)} w.r.t. $\underline{S}$

Given bids, how find prices & allocations?

Extending Bank of England solution to -ve bids

(alternative method: paper with Baldwin, Goldberg, Lock)

"Strong substitutes" preference structure allows a variant of steepest descent method for discrete functions (*cf*. Kelso and Crawford (1982), Murota and co-authors, Milgrom (2000), Ausubel (2006))

Standard steepest descent "crawls" through price grid, but using geometry of bids turns "crawling" into "leaping"

"Arctic" auction (programmed for Iceland & for IMF)

(algorithms developed with help of Fichtl, Lock, dotEcon)

cf. linear Fisher market; Eisenberg-Gale algorithms, etc.

Product-Mix Auction offers lots more options:

Alternative **forms of preferences**,

e.g., BoE's *total* supply now depends on bidding

Alternative Pricing Rules

(including Pay-as-bid pricing)

Alternative Auctioneer's Objective**:**

e.g., BoE: efficiency *vs.* Iceland: minimise cost

Permit multiple sellers (as well as multiple buyers)

&/or *traders* (who can both buy and sell)

Summary

Geometry is useful for designing, representing, and analysing bidding languages

Bidders want to express different information in different contexts (substitutabilities, complementarities, budget constraints, etc.)

→ need to offer different languages

Trade off --expressivity

--ease of use and understanding

--ability to analyse and solve auctions

Need both economic and computer-scientific understanding

For more information see my website

especially "Product-Mix Auctions" (working paper)