Model for the auction game

Suppose that the true value S is drawn uniformly from $[0,1]$. Then, conditional on S, every bidder observes $X_i = S\Delta_i$, where Δ_i is drawn i.i.d. uniformly from $[0,2]$. There are N bidders. We want to compute the equilibrium bidding function in sealed-bid second- and first-price auctions, relying on the derivations in Milgrom and Weber (1982). [Please note that this is not the same game that was played during the workshop; first, because no payments were actually charged to bidders in the auction games, our auctions were simulated auctions, and second, information sharing was allowed.]

Calculations of relevant objects

Throughout, we use F and f, with subscripts, to denote the CDF and PDF, respectively, of the random variables in question.

In our model, we have

$$F_{X_i|S}(x|s) = \mathbb{P}(S\Delta_i \le x|S=s) = \max\left\{\frac{x}{2s}, 1\right\},$$

$$f_{X_i|S}(x|s) = \frac{1}{2s}\mathbf{1}_{\{x\le 2s\}}.$$

From that, we can calculate

$$f_{S|X_i}(s|x) = \frac{f_{X_i|S}(x|s)f_S(s)}{\int f_{X_i|S}(x|s)f_S(s)ds} = \frac{\frac{1}{s}\mathbf{1}_{\{x\le 2s\}}}{\int_{\frac{x}{2}}^{1}\frac{1}{s}ds} = \frac{\frac{1}{s}\mathbf{1}_{\{x\le 2s\}}}{\log(2)-\log x}.$$

Therefore,

$$\mathbb{E}[S|X_i=x] = \int s f_{S|X_i}(s|x)ds = \int_{\frac{x}{2}}^{1}\frac{1}{\log(2)-\log x}ds = \frac{1-\frac{x}{2}}{\log(2)-\log x}.$$

This is the bidder's estimate of the true value based on her private signal realization x, before accounting for the learning in the auction.

Let Y_i be the highest signal among all bidders other than i: $Y_i = \max_{j\ne i} X_j$. We have that

$$F_{Y_i|S}(y|s) = \mathbb{P}(S(\max_{j\ne i}\Delta_j) \le y|S=s) = \left[\mathbb{P}(S\Delta_i\le y|S=s)\right]^{N-1} = \max\left\{\left(\frac{y}{2s}\right)^{N-1}, 1\right\},$$

$$f_{Y_i|S}(y|s) = (N-1)\left(\frac{y}{2s}\right)^{N-2}\frac{1}{2s}\mathbf{1}_{\{y\le 2s\}}.$$

Note that we have

$$f_{S|X_i,Y_i}(s|x,y) = \frac{f_{Y_i|S}(y|s)f_{X_i|S}(x|s)f_S(s)}{\int f_{Y_i|S}(y|s)f_{X_i|S}(x|s)f_S(s)ds} = \frac{\left(\frac{1}{s}\right)^N\mathbf{1}_{\left\{s\ge\max\left\{\frac{x}{2},\frac{y}{2}\right\}\right\}}}{\int_{\max\left\{\frac{x}{2},\frac{y}{2}\right\}}^{1}\left(\frac{1}{s}\right)^N ds}$$

$$= \frac{(N-1)s^{-N}\mathbf{1}_{\left\{s\ge\max\left\{\frac{x}{2},\frac{y}{2}\right\}\right\}}}{\left[\left(\max\left\{\frac{x}{2},\frac{y}{2}\right\}\right)^{1-N}-1\right]},$$

and hence

$$\mathbb{E}[S|X_i = x,\, Y_i = y] = \int s f_{S|X_i,Y_i}(s|x,y)ds = \frac{N-1}{\left(\max\left\{\frac{x}{2},\, \frac{y}{2}\right\}\right)^{1-N} - 1} \int_{\max\left\{\frac{x}{2},\, \frac{y}{2}\right\}}^{1} s^{1-N} ds$$

$$= \frac{N-1}{N-2} \frac{\left(\max\left\{\frac{x}{2},\, \frac{y}{2}\right\}\right)^{2-N} - 1}{\left(\max\left\{\frac{x}{2},\, \frac{y}{2}\right\}\right)^{1-N} - 1}$$

Finally, we want to compute

$$f_{Y_i|X_i}(y|x) = \int f_{Y_i|S}(y|s) f_{S|X_i}(s|x)ds = (N-1)\frac{2y^{N-2}(2)^{-N}}{\log(2) - \log x} \int_{\max\left\{\frac{x}{2},\, \frac{y}{2}\right\}} s^{-N} ds$$

$$= \frac{y^{N-2} 2^{1-N}}{\log(2) - \log x} \left[\left(\max\left\{\frac{x}{2},\, \frac{y}{2}\right\}\right)^{1-N} - 1\right],$$

and hence

$$F_{Y_i|X_i}(y|x) = \frac{1}{N-1} y^{N-1} \frac{2^{1-N}}{\log(2) - \log x} \left[\left(\frac{x}{2}\right)^{1-N} - 1\right],$$

for $y \leq x$. So now we have

$$\frac{f_{Y_i|X_i}(x|x)}{F_{Y_i|X_i}(x|x)} = \frac{x^{-2}}{\frac{1}{N-1} x^{-1}} = \frac{N-1}{x}.$$

$$L(\alpha|x) := \exp\left(-\int_{\alpha}^{x} \frac{f_{Y_i|X_i}(s|s)}{F_{Y_i|X_i}(s|s)} ds\right) = \exp\left((N-1)(\log(\alpha) - \log(x)\right) = \left(\frac{\alpha}{x}\right)^{N-1}.$$

This function will be needed to compute the equilibrium of the FPA.

SPA

Based on Theorem 6 in MW'82, the equilibrium bidding function in a SPA is (for signal x)

$$b^{\star}(x) = \mathbb{E}[S|X_i = x,\, Y_i = x] = \frac{N-1}{N-2} \frac{\left(\frac{x}{2}\right)^{2-N} - 1}{\left(\frac{x}{2}\right)^{1-N} - 1}.$$

For large N, this is approximately $\frac{N-1}{N-2}\frac{x}{2}$.

FPA

Based on Theorem 14 in MW'82, the equilibrium bidding function in a FPA is (for signal x):

$$b^{\star}(x) = \int_0^x \mathbb{E}[S|X_i = \alpha,\, Y_i = \alpha] dL(\alpha|x) = \int_0^x \frac{N-1}{N-2} \frac{\left(\frac{\alpha}{2}\right)^{2-N} - 1}{\left(\frac{\alpha}{2}\right)^{1-N} - 1} (N-1) \left(\frac{\alpha}{x}\right)^{N-2} \frac{1}{x} d\alpha.$$

Using the above approximation, we can write

$$\int_0^x \frac{N-1}{N-2} \frac{\alpha}{2} (N-1) \left(\frac{\alpha}{x}\right)^{N-2} \frac{1}{x} d\alpha \approx \frac{N-1}{N-2} \frac{N-1}{2} \left(\frac{1}{x}\right)^{N-1} \int_0^x \alpha^{N-1} d\alpha$$

$$= \frac{N-1}{N-2} \frac{N-1}{2} \left(\frac{1}{x}\right)^{N-1} \frac{1}{N} x^N = \frac{(N-1)^2}{N(N-2)} \frac{x}{2}.$$

Some comments

Note that for large N, it is an approximately optimal strategy in both auctions to bid half of the signal realization. In particular, the optimal bids in both auction formats are approximately the same when there are many bidders. The figures below illustrate these findings by plotting the estimate of the value based on the private signal, and the optimal bidding functions for the cases $N = 5$ and $N = 50$ (everything is multiplied by 100 to convert to the numbers in the auction game during the workshop).