Solution to Exchanges 10.2 Puzzle: Borrowing in the Limit as our Nerdiness Goes to Infinity

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This is a solution to the editor's puzzle from issue 10.2 of SIGecom Exchanges [Reeves 2011]. The Puzzle asks to determine a point in time such that a lump sum payment of $S will be equivalent to a continuous stream of infinitesimal payments totaling $S, spread evenly over time. The full puzzle can be found online at: http://www.sigecom.org/exchanges/volume_10/2/puzzle.pdf.

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The puzzle asks to determine a point in time such that a lump sum payment of $S will be equivalent to a continuous stream of infinitesimal payments totaling $S, spread evenly over an amount of time T. This note presents a short solution and some of my intuitions.[1] In the spirit of the puzzle, I start with an illustration of the nature of the annualized interest rate.

Let r be the annualized interest rate. If the rate of interest is fixed, the semi-annual rate r_2 is such that $(1+r_2)^2 = 1+r$. If I save a dollar for a year, I will then earn r_2 dollars after the first half of the year. I will earn more money in the second half of the year, because I will begin the period with a higher balance. This is the miracle of compound interest. For a general n, the rate of interest for $\frac{1}{n}$ of a year r_n, is such that $(1+r_n)^n = 1+r$, hence:

$$n \log(1+r_n) = \log(1+r)$$

Let $\tilde{r} := \log(1+r)$. One can observe easily that the interest rate for time $\frac{k}{n}$ is $e^{\tilde{r}\frac{k}{n}} - 1$, and a continuity argument (or a different partition to sub-periods) implies that the interest rate until t is $e^{\tilde{r}t}-1$. I shall therefore refer to $\tilde{r}$ as the instantaneous rate of interest.

I now turn to solve the puzzle, for the case S=$1 , and T=1 year. A continuous (flow) payment of $1 per year discounted to current value is worth:

$$\int_0^1 e^{-\tilde{r}t} dt = \frac{1-e^{-\tilde{r}}}{\tilde{r}} \tag{1}$$

[1]The author thanks Daniel Reeves and Assaf Romm for helpful comments and suggestions.

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The discounted value of a one time payment of $1 at time t is $e^{-\tilde{r}t}$. Specifically, for $t = 0$ this expression equals 1, and for $t = 1$ it is $e^{-\tilde{r}} = \frac{1}{1+r}$. Hence, for the case of $1 over one year, the solution for the puzzle is the solution of $e^{-\tilde{r}t} = \frac{1-e^{-\tilde{r}}}{\tilde{r}} \Rightarrow t = -\frac{1}{\tilde{r}} \log \frac{1-e^{-\tilde{r}}}{\tilde{r}}$. Solving for different time periods or sums of money is just a re-parametrization of the current problem.[2] An alternative proof uses the limit of the discrete process described in the puzzle. First, calculate the current value of n equally spread payments of $\frac{1}{n}$:

$$\frac{1}{n}\sum_{j=0}^{n-1} \frac{1}{(1+r)^{\frac{j}{n}}} = \frac{1}{n} \frac{1 - \frac{1}{1+r}}{1 - (\frac{1}{1+r})^{\frac{1}{n}}} = \frac{r}{1+r} \frac{\frac{1}{n}}{1 - (\frac{1}{1+r})^{\frac{1}{n}}} \underset{n\to\infty}{\to} \frac{r}{1+r} \frac{-1}{\log(\frac{1}{1+r})} \tag{2}$$

A payment of 1 at t has the current value $\frac{1}{(1+r)^t}$. So we are looking for a solution for:

$$\frac{1}{(1+r)^t} = \frac{r}{1+r} \frac{-1}{\log(\frac{1}{1+r})} \tag{3}$$

Rearranging 3 yields:

$$(1-t)\log(1+r) = \log\left(\frac{-r}{\log(\frac{1}{1+r})}\right) \tag{4}$$

Recall that $e^{-\tilde{r}} = \frac{1}{1+r} \Rightarrow -\tilde{r} = \log \frac{1}{1+r}$, $e^{\tilde{r}} - 1 = r$, and $\log(1+r) = \tilde{r}$. Substitution yields:

$$(1-t)\tilde{r} = \log \frac{1-e^{\tilde{r}}}{-\tilde{r}} \Rightarrow 1 - t = \frac{1}{\tilde{r}} \log \frac{1-e^{\tilde{r}}}{-\tilde{r}} \Rightarrow$$

$$t = 1 - \frac{1}{\tilde{r}} \log \frac{1-e^{\tilde{r}}}{-\tilde{r}} = \frac{1}{\tilde{r}}(\log e^{\tilde{r}} - \log \frac{1-e^{\tilde{r}}}{-\tilde{r}}) = -\frac{1}{\tilde{r}} \log \frac{1-e^{\tilde{r}}}{-\tilde{r}e^{\tilde{r}}} = -\frac{1}{\tilde{r}} \log \frac{1-e^{-\tilde{r}}}{\tilde{r}}$$

the same solution we got before.

It is interesting to observe that $\lim_{r\to\infty} t = 0$ and $\lim_{r\to 0^+} t = \frac{1}{2}$. That is, when the interest rate is low (and the horizon, T, is short), the payment should be made near the mid-point.[3] When the interest rate is high (and the horizon, T, is long), the effect of compounding is large, and so the fair payment timing is early. For example, when T=1 year and r is 10% the fair payment time is close to .496. To see why t approaches 0 as r grows to infinity, imagine that r is very large. After 1% of the period had passed, 1% of the money must have been paid in the continuous payment scheme. With r large enough, leaving this money in the bank for 1% of the period would yield more than $S. Therefore, the fair t must be earlier than 2% of the period.

Finally, the puzzle assumed that nerdiness is of the type we are used to - nerds who like complicated calculations. However, one can imagine a world where agents

[2]The solution remains the same for different values of S. For different T, r should be replaced with the interest over T years, and the solution should be interpreted as a proportion out of T.
[3]Note that when $r = 0$ the timing doesn't matter, and any t solves the problem.

prefer to avoid the variation in the timing of the lump sum payoff, and find it difficult to pay continuously. Equation 1 offers a simple solution for such agents. An infinitesimal (flow) payment of \$1 would yield the same profit as an interest free loan of $\$\frac{1}{\tilde{r}}$ for the same duration. Moreover, the solution suggests a simple compensation scheme for the provision of continuous services; $\$\frac{S}{\tilde{r}}$ are deposited in an escrow account and kept in the account for the duration. Then, the interest accrued is paid to to the service provider and the principal is refunded to the client.

REFERENCES

Reeves, D. 2011. Borrowing in the limit as our nerdiness goes to infinity. *SIGecom Exch. 10,* 2 (June), 50–50.