Do Prices Coordinate Markets?

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A Walrasian equilibrium outcome has a remarkable property: the induced allocation maximizes social welfare while each buyer receives a bundle that maximizes her individual surplus at the given prices. There are, however, two caveats. First, minimal Walrasian prices necessarily induce indifferences. Thus, without coordination, buyers may choose surplus maximizing bundles that conflict with each other. Accordingly, buyers may need to coordinate with one another to arrive at a socially optimal outcome—the prices alone are *not* sufficient to coordinate the market. Second, although natural auctions converge to Walrasian equilibrium prices on a fixed population, in practice buyers typically observe prices without participating in a price computation process. These prices are not perfect Walrasian equilibrium prices, but we may hope that they still encode distributional information about the market. To better understand the performance of Walrasian prices in light of these two problems, we give two results. First, we propose a mild *genericity* condition on valuations so that the minimal Walrasian equilibrium prices induce allocations resulting in low overdemand, no matter how the buyers break ties. In fact, under our condition the overdemand of any good can be bounded by 1, which is the best possible at the minimal prices. Second, we use techniques from learning theory to argue that the overdemand and welfare induced by a price vector converge to their expectations uniformly over the class of all price vectors, with sample complexity linear and quadratic in the number of goods in the market respectively. The latter results make no assumption on the form of the valuation functions.

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> *"Fundamentally, in a system in which the knowledge of the relevant facts is dispersed among many people, prices can act to coördinate the separate actions of different people in the same way as subjective values help the individual to coördinate the parts of his plan... We must look at the price system as such a mechanism for communicating information..."*
>
> — Friedrich A. Hayek, The Use of Knowledge in Society

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1. INTRODUCTION

The ability of prices to both encode preferences and signal the scarcity or abundance of a resource is a cynosure of the neo-classical tradition in economics. A particularly well-studied from of pricing is an *anonymous item* pricing when each item has a constant unit price available to all buyers, and buyers may buy arbitrary bundles of resources. If for some such pricing $\vec{p} = (p_1, \ldots, p_k)$ every buyer can purchase a bundle maximizing her utility at $\vec{p}$ and all resources are sold, then $\vec{p}$ are called *Walrasian* prices and the resulting allocation is called a *Walrasian* allocation. It is known that any Walrasian prices support any Walrasian allocation, and that all Walrasian allocations maximize social welfare. There are even decentralized, polynomial-time algorithms for computing Walrasian prices and allocations [Kelso and Crawford 1982].

When a Walrasian equilibrium (a pair of Walrasian prices and corresponding allocation) exists, we can think of the prices as *coordinating* the purchasing decisions of the buyers in the market, since each buyer can individually purchase her favorite bundle of goods while clearing the market and achieving high social welfare. Unfortunately, there are two problems with this interpretation. First, buyers are often indifferent between several bundles of most-preferred goods, and if they do not coordinate their *tie-breaking* then the demand of a good may exceed its supply, leading to an infeasible allocation. Second, while Walrasian prices clear the market for a fixed set of buyers, in practice prices are often not calculated with a fixed set of buyers in mind. In practice, prices are usually set based on more general data about the buyers, and later applied to random samples of buyers. Our work [Hsu et al. 2016] addresses both shortcomings in the "story" of prices coordinating buyers in a decentralized market.

2. UNCOORDINATED TIE-BREAKING AND BUYER GENERICITY

We first discuss the problem, and our solution, to buyer indifferences at equilibrium. In a nutshell, if buyers face multiple favorite bundles at the equilibrium prices and do not coordinate their tie-breaking, they may jointly demand an allocation with poor social welfare or, worse, an infeasible allocation. In fact, for the *minimal* Walrasian prices, it is not hard to see that there must *always* be buyers who are indifferent between two bundles of goods.

Furthermore, it is not hard to cook up a worst-case example where a good's demand might exceed its supply by $\Omega(n)$, where n is the number of buyers in the market. Consider a market where every buyer is identical: in such a market, any most-preferred bundle for one buyer is a most-preferred bundle for every buyer! See Fig. 1 for a picture of one bad example.

If similar buyers all participate in a market, it is clear that some amount of coordination beyond the announcing of Walrasian prices will be necessary to reach a feasible allocation. This problem motivates our definition of *genericity* of a set of buyers, which guarantees that buyers are not too similar by requiring that their valuations are linearly independent. If the set of buyers in a market are generic, we show that uncoordinated tie-breaking at minimal Walrasian prices achieves a fairly good allocation: regardless of how buyers choose their most-preferred bundle, every good will be demanded by at most *one* buyer beyond the supply of that good.

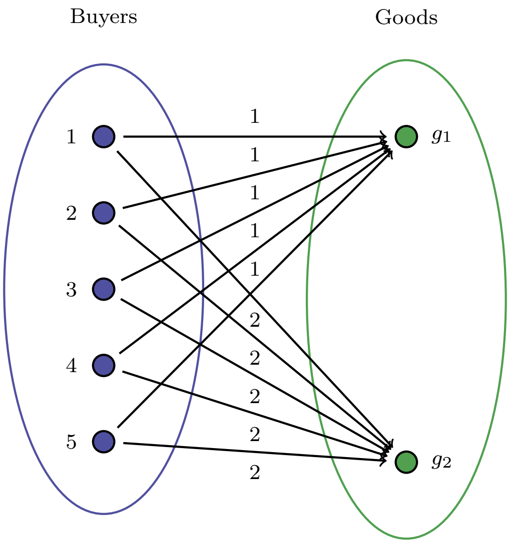

Fig. 1. Suppose buyers are unit-demand (their value for a bundle of goods is just their value for the item in that bundle they desire the most). If buyers all have identical values for each good, as here, then at any prices, all buyers will have identical sets of most-preferred bundles.

Below, we give the formal definition of genericity when buyers have unit-demand valuations. The definition of genericity for more general valuations can be found in the extended version of this work [Hsu et al. 2016].

Definition 2.1 *Generic valuations.* A set of unit-demand valuations $\{v_{g,i} \in \mathbb{R} : i \in [n], g \in [k]\}$ is *generic* if they are linearly independent over $\{-1, 0, 1\}$, i.e.

$$\sum_{q \in N} \sum_{g \in [k]} \alpha_{i,g} v_{g,i} = 0 \qquad \text{for } \alpha_{g,i} \in \{-1, 0, 1\}$$

$$\text{implies } \alpha_{g,i} = 0 \text{ for all } i \in [n], g \in [k].$$

Informally, generic buyers are not too similar. For instance, any two buyers may not have identical valuations. This condition is actually more restrictive that just this property; it rules out various pathological examples with high overdemand. In fact, the genericity condition almost completely eliminates the overdemand problem: if buyers have generic valuations, they can break ties *arbitrarily*, without coordinating, amongst their most-demanded sets at the minimal Walrasian prices without causing overdemand more than 1.

THEOREM 2.2. *When buyers are generic*[1] *and arbitrarily choose amongst their most-preferred bundles at the minimal Walrasian prices, the overdemand of any good is at most* 1.

[1]More precisely, we assume that buyers have a generic version of *matroid-based valuations*, a class of valuations hypothesized to be equivalent to gross substitute valuations [Ostrovsky and Paes Leme 2015].

Since overdemand is at least 1 at the minimal prices—there must be some indifferences—Theorem 2.2 shows that the genericity condition achieves the best overdemand we can hope for. Our results hold regardless of the supply of each good. In particular, if we think of the supply of each good as growing with the number of buyers, the 1 unit of overdemand is a vanishing fraction of the overall supply.

3. STATISTICAL GENERALIZABILITY OF WALRASIAN PRICES

We now discuss the second problem in the story of Walrasian prices coordinating markets: prices are often set using a *sample* of buyers from a market, and applied to a second sample, with the hope that the prices will lead to similar welfare and demand on the two samples. In learning-theory terminology, we can think of the first group of buyers as the *training* sample, and the second group of buyers are the *test* sample.

We show that the welfare and demand induced by a fixed set of prices are well-behaved quantities in a statistical sense.[2] If two samples of buyers (the *training* and the *test* samples) are drawn from the same distribution $\mathcal{D}$ and the market is large enough, then the Walrasian prices for the training sample will be approximately Walrasian for the test sample, assuming the same supply of each good. This result follows from two claims: first, using Walrasian prices $\vec{p}$ computed on the training set, that the demand from the test sample for each good will be approximately equal to its supply; second, that $\vec{p}$ induces approximately optimal welfare on the test sample.

First, we show that the Walrasian prices calculated on a training sample of buyers induce approximately optimal welfare on a fresh sample of buyers. In order for this to hold, it must be that the *expected* optimal welfare is reasonably large.

THEOREM 3.1. *Suppose S and S' are two sets of buyers of size n, each drawn independently from a distribution over valuations $\mathcal{D}$. Then if p are the minimal Walrasian prices for S, and we have*

$$\mathbb{E}\left[\text{OPT-}\textsc{Welfare}\right] \geq \tilde{O}\left(\frac{k^4\sqrt{n}\ln^2\frac{1}{\delta}}{\alpha^2}\right),$$

for $\alpha \in (0, \frac{4}{5})$, then $\textsc{Welfare}(p, S') \geq (1-\alpha)\text{OPT-}\textsc{Welfare}(S')$ *with probability $1-\delta$.*

Then, we show that the Walrasian prices for S approximately clear the market for buyers S'.

THEOREM 3.2. *Suppose $|S| = |S'| = n$ and $v^i \sim \mathcal{D}$ for all $v^i \in S \cup S'$. Then if $\vec{p}$ are the minimal Walrasian prices for S,*

$$\textit{Supply of } g \geq \tilde{O}\left(\frac{k\ln\frac{1}{\delta}}{\alpha^2}\right),$$

[2] Similar results were previously known for welfare [Balcan et al. 2008; Devanur and Hayes 2009; Agrawal et al. 2014], and demand [Agrawal et al. 2014], although previous results for demand require the supply of each good to be *quadratic* in the number of types of goods, rather than linear, as we show here.

for $\alpha \in (0, \frac{4}{5})$ *and for each good* $g \in [k]$*, then for each* g *with* $\vec{p}_g > 0$*,*

$$(1-\alpha)\,[\textit{Supply of } g] \leq \textit{Demand for } g \textit{ from } S' \leq (1+\alpha)\,[\textit{Supply of } g]$$

with probability $1 - \delta$*.*

These results demonstrate that Walrasian prices generalize in a strong sense: Walrasian prices for one market will be (approximately) Walrasian for an identically distributed but different market. Theorems 3.1 and 3.2 are proved by bounding the VC-dimension and pseudo-dimension (a real-valued analogue of VC dimension) of the class of pricings and leveraging uniform convergence results from statistical learning theory.

4. OPEN QUESTIONS

Our work leaves several intriguing questions. First, does our genericity result Theorem 2.2 extend to richer classes of valuations? Second, our uniform convergence of welfare results in Theorem 3.1 use a loose bound on the pseudo-dimension of prices, which depends quadratically the number of kinds of goods in the market, while the demand generalization has only linear dependence. Is this quadratic dependence necessary? Also, is quartic dependence on the number of kinds of goods necessary for uniform concentration of welfare?

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