Approximating Nash Social Welfare under Rado Valuations

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The Nash social welfare problem asks for an allocation of indivisible items to agents in order to maximize the geometric mean of agents' valuations. We give an overview of the constant-factor approximation algorithm for the problem when agents have Rado valuations [Garg et al. 2021]. Rado valuations are a common generalization of the assignment (OXS) valuations and weighted matroid rank functions. Our approach also gives the first constant-factor approximation algorithm for the asymmetric Nash social welfare problem under the same valuations, provided that the maximum ratio between the weights is bounded by a constant.

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1. INTRODUCTION

Given a set $\mathcal{G}$ of m indivisible items and a set $\mathcal{A}$ of n agents with valuations $v_i : 2^{\mathcal{G}} \to \mathbb{R}_+$, we need to allocate the items to the agents in a fair and efficient manner. Different allocations $(S_1, \ldots, S_n)$ will result in different values $v_i(S_i)$ for the agents. How should we allocate the items to balance fairness and efficiency?

A common measure of efficiency is maximizing the *utilitarian social welfare*, i.e., finding an allocation $(S_1, \ldots, S_n)$ that maximizes $\sum_{i\in\mathcal{A}} v_i(S_i)$. Naturally, efficiency comes at the expense of fairness: in an optimal utilitarian social welfare allocation some agents might receive no items.

On the other side, maximizing fairness is often associated with maximizing the minimum value across all agents, i.e., $\max_{(S_1,S_2,\ldots,S_n)} \min_{i\in\mathcal{A}} v_i(S_i)$. This is also known as *max-min fairness* or the *Santa Claus* problem. Fairness comes at the expense of efficiency: we might have to assign most of the items to an agent with low valuation of all subsets of $\mathcal{G}$ when compared to the valuations of other agents.

An objective that lies between fairness and efficiency is the Nash social welfare, defined for an allocation $(S_1, \ldots, S_n)$ as the geometric mean of agents' valuations.

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The corresponding (symmetric) NSW problem is

$$\max\left\{\left(\prod_{i\in\mathcal{A}} v_i(S_i)\right)^{1/n} : \{S_i\}_{i\in\mathcal{A}} \text{ is a partition of } \mathcal{G}\right\} .$$

A distinctive feature of the NSW objective is invariance under scaling of the valuations. That is, unlike the utilitarian social welfare and the max-min fairness, the set of optimal allocations in the NSW problem remains unchanged even if the valuations of the agents are scaled by arbitrary positive constants.

More generally, in our paper [Garg et al. 2021], we study the asymmetric NSW problem in which agents might have different importance. Here, next to the valuation $v_i : 2^{\mathcal{G}} \to \mathbb{R}_+$, each agent i is also given a weight w_i and the goal is to maximize the weighted geometric mean of agents' valuations, i.e.,

$$\max\left\{\left(\prod_{i\in\mathcal{A}} v_i(S_i)^{w_i}\right)^{1/\sum_{i\in\mathcal{A}} w_i} : \{S_i\}_{i\in\mathcal{A}} \text{ is a partition of } \mathcal{G}\right\} . \tag{NSW}$$

Origins. The Nash social welfare is a natural concept that was discovered independently as the unique solution to a bargaining game [Nash 1950; Kaneko and Nakamura 1979], competitive equilibrium with equal incomes [Varian 1974], and proportional fairness in networking [Kelly 1997]. These papers considered the symmetric NSW objective. The asymmetric objective has also been well-studied since the seventies [Harsanyi and Selten 1972; Kalai 1977] with applications in bargaining theory, water resource allocation, and climate agreements.

Previous work. From a computational perspective, the NSW problem is NP-hard already for additive valuations. For the symmetric NSW problem with additive valuations a variety of methods have been used. The seminal paper by Cole and Gkatzelis [2018][1] gave the first constant-factor approximation algorithm by rounding a *spending-restricted* market equilibrium. This was followed by a constant-factor approximation algorithm built on the theory of *real stable polynomials* [Anari et al. 2017]. The state-of-the-art approximation factor is 1.45 based on *local search* [Barman et al. 2018]. These approaches have been extended to constant-factor approximation algorithms for mild generalizations of additive valuations [Garg et al. 2018; Anari et al. 2018; Chaudhury et al. 2018; Garg et al. 2021].

2. OUR MAIN RESULT

Our main focus is on the following subclass of submodular valuations. We propose the name "Rado valuations" in honor of Richard Rado, who first studied the independent matching problem [Rado 1942].

Definition 2.1. For the set of items $\mathcal{G}$, we consider a bipartite graph $(\mathcal{G}, V; E)$ with a cost function $c : E \to \mathbb{R}_+$ on the edges, and a matroid $\mathcal{M} = (V, \mathcal{I})$. For a subset of items $S \subseteq \mathcal{G}$, the *Rado valuation* $v(S)$ is defined as the maximum cost of a matching M in $(\mathcal{G}, V; E)$ such that $\delta_{\mathcal{G}}(M) \subseteq S$ and $\delta_V(M) \in \mathcal{I}$, i.e.,

$$v(S) := \max\left\{\sum_{e\in M} c(e) : M \text{ is a matching}, \delta_{\mathcal{G}}(M) \subseteq S, \delta_V(M) \in \mathcal{I}\right\} .$$

Our main contributions are a constant-factor approximation algorithm for the symmetric NSW problem under Rado valuations, and a constant-factor approxima-

[1] Also discussed in SIGecom Exchanges [Cole and Gkatzelis 2015].

tion algorithm for asymmetric NSW provided the ratios between the weights are bounded. Namely, our approximation factors depend on γ, for $\gamma := 1 + \max_{i\in\mathcal{A}} w_i$.

THEOREM 2.2. *There is a polynomial-time* $256e^{3/e}\approx 772$*-approximation algorithm for the symmetric Nash social welfare problem under Rado valuations. For the asymmetric problem, there is a* $256\gamma^3$*-approximation algorithm for Rado valuations, and a* 16γ*-approximation algorithm for additive valuations.*

Our algorithm is based on a careful rounding of a mixed integer programming relaxation in multiple stages. An enticing feature of our algorithm is modularity: out of the five phases, only one requires properties of Rado valuations, and all other steps work for general subadditive valuations, assuming they are given with a suitable convex extension.

We note that even if the weights of the agents are bounded, an $O(1)$-approximation for the symmetric case does not yield an $O(1)$-approximation to the asymmetric case. To illustrate this point, consider two items $\{a, b\}$ and two agents with weights $w_1 = 1$, $w_2 = 2$ and additive valuations $v_1(a) = M + 1$, $v_1(b) = 1$, $v_2(a) = M$, $v_2(b) = 1$ for $M \gg 1$. The unique optimal solution to the symmetric case ($w'_1 = w'_2 = 1$) allocates a to agent 1 and b to agent 2. However, this returns an NSW value $(M + 1)^{1/3}$ for the original weights. This is worse by factor $\approx M^{1/3}$ than the NSW value $M^{2/3}$ obtainable by assigning b to 1 and a to 2.

Li and Vondrák recently contributed two exciting developments in this context. In [2021b], they gave a $\frac{e^3}{(e-1)^2}$-approximation of the optimum Nash social welfare value for a broad class of submodular valuations, including the cone generated by Rado valuations. Even though this is a broader class and the approximation ratio is significantly better, the paper does not yield a polynomial-time algorithm to find a corresponding allocation. The approach is based on real stable polynomials.

Subsequently, in [2021a] they gave a significant extension of our paper to obtain a 380-approximation algorithm for *arbitrary* monotone submodular valuations.

3. RADO VALUATIONS: CONTEXT AND EXAMPLES

In the simplest case when $\mathcal{M}$ is the free matroid on V, i.e., $\mathcal{I} = 2^V$, the value of a set S is the maximum cost matching in the subgraph induced by $S \cup V$. Such valuations are known as *assignment valuations* or *OXS valuations*.

Shapley [1962] gave a nice interpretation of assignment valuations. Assume that each agent is a company. Furthermore, assume that the items $\mathcal{G}$ are workers and V is the set of jobs within a particular company. The edge set represents the possibilities (willingness) of assigning workers to jobs, and the cost c_{jk} is value the company gets by assigning worker j to job k. By the definition of assignment valuations, the value of a subset $S \subseteq \mathcal{G}$ of workers for the company is the maximum possible value the company gets by assigning workers S to jobs V.

The same interpretation extends to Rado valuations with the additional possibility that the occupied set of jobs must be an independent set in matroid $\mathcal{M}$. E.g., the company may partition the set of all jobs V into certain types, and require that at most one job of each type to be assigned—a partition matroid constraint.

As another example of Rado valuations, consider the case where V is a copy of the set of items $\mathcal{G}$, with each $j \in \mathcal{G}$ having a corresponding $j' \in V$, and let

$E = \{(j, j') : j \in \mathcal{G}\}$. Let $g : \mathcal{G} \to \mathbb{R}$, and $c_{jj'} = g_j$ for all $j \in \mathcal{G}$, and let r be rank function of $\mathcal{M}$. In this case the $v(S)$ is a weighted matroid rank function (WMR).

Relations between popular classes of valuations are:

$$\text{Additive} \subsetneq \text{SPLC} \subsetneq \begin{matrix}\text{OXS}\\ \text{WMR}\end{matrix} \subsetneq \text{Rado} \subsetneq \text{GS} \subsetneq \text{Submodular} \subsetneq \text{Subadditive}\,.$$

Construction of substitutes. Assignment valuations and weighted matroid rank functions are well-known examples of gross substitutes valuations (GS). This is true for all Rado valuations. Frank asked in 2003 whether the converse is also true: is the class of gross substitute valuations the same as those of Rado valuations? We show that this is not the case. The reason is that, unlike gross substitute, Rado valuations are not closed under *endowment operation*.

For $v : 2^{\mathcal{G}} \to \mathbb{R}$ and $T \subseteq \mathcal{G}$, we define the *endowed* valuation $v' : 2^{\mathcal{G}' \setminus T} \to \mathbb{R}_+$ as $v'(X) = v(X \cup T) - v(T)$. Endowment can be seen as a minor operation. We say that v is a Rado minor valuation if it is an endowed Rado valuation. We propose a natural refinement of the conjecture that also generalizes *matroid based valuations* conjecture [Ostrovsky and Paes Leme 2015].

CONJECTURE 3.1. *Every gross substitutes valuation is a Rado minor valuation.*

4. MAIN STEPS OF OUR APPROACH

The valuations in the NSW problem are defined on subsets of $\mathcal{G}$. Any arguments based on convex relaxations require a continuous (concave) extension of the valuations to $\mathbb{R}^{\mathcal{G}}_+$. We provide such an extension for Rado valuations. Here, $\nu : [0,1]^{\mathcal{G}} \to \mathbb{R}_+$ is a concave extension of $v : 2^{\mathcal{G}} \to \mathbb{R}_+$ if ν is concave, and ν and v take the same value on integer points. Concave extensions exist for GS valuations but not for submodular valuations in general.

THEOREM 4.1. *Let $v : 2^{\mathcal{G}} \to \mathbb{R}_+$ be a Rado valuation, and let r be the rank function of $\mathcal{M}$. The concave extension of v is defined for $x \in [0,1]^{\mathcal{G}}$ as*

$$\begin{aligned} v(x) := \quad \max \quad & \textstyle\sum_{(j,k)\in E} c_{jk} z_{jk} \\ \textit{s.t.:} \quad & \textstyle\sum_{k\in V} z_{jk} \le x_j \qquad \forall j \in \mathcal{G} \\ & \textstyle\sum_{j\in\mathcal{G},k\in T} z_{jk} \le r(T) \qquad \forall T \subseteq V \\ & z \ge 0\,. \end{aligned} \tag{Rado ext}$$

Now, we can relax the initial problem. The natural relaxation of (NSW) has unbounded integrality gap already for additive valuations. Instead, we propose a mixed integer programming relaxation. For a set of items $\mathcal{H} \subseteq \mathcal{G}$, we consider the relaxation where $\mathcal{H}$ is allocated integrally and the rest fractionally.

$$\begin{aligned} \max \quad & \left(\textstyle\prod_{i\in\mathcal{A}} v_i(x_i)^{w_i}\right)^{1/\sum_i w_i} \\ \text{s.t.:} \quad & \textstyle\sum_{i\in\mathcal{A}} x_{ij} \le 1 \qquad \forall j \in \mathcal{G} \\ & x_{ij} \in \{0,1\} \qquad \forall j \in \mathcal{H}, \forall i \in \mathcal{A} \\ & x \ge 0\,. \end{aligned} \tag{Mixed relaxation}$$

Our algorithm will construct a set $\mathcal{H}$ and an allocation $x \in \{0,1\}^{\mathcal{A}\times\mathcal{G}}$ such that x is $256\gamma^3$-approximate solution of (Mixed relaxation). This is proved in five phases.

4.1 **Phase I**: Finding the item set $\mathcal{H}$

We find a matching τ with the highest NSW. This can be achieved by solving the maximum weight assignment problem in the complete bipartite graph between $\mathcal{A}$ and $\mathcal{G}$ with edge weights $\omega_{ij} = w_i \log(v_i(j))$ for every $i \in \mathcal{A}$, $j \in \mathcal{G}$. Let $\tau : \mathcal{A} \to \mathcal{G}$ be an optimal matching represented as a mapping, i.e., $\tau(i)$ is the item matched to agent $i \in \mathcal{A}$. We define $\mathcal{H} := \tau(\mathcal{A})$, i.e., $\mathcal{H}$ is the set of the items assigned by τ.[2]

4.2 **Phase II**: Reduction to the mixed matching relaxation

We approximate (Mixed relaxation) by a second mixed integer program.

$$\begin{aligned} \max \quad & \left(\prod_{i\in\mathcal{A}} \left(v_i(y_i) + v_{i\sigma(i)}\right)^{w_i}\right)^{1/\sum_i w_i} && \\ \text{s.t.:} \quad & \sum_{i\in\mathcal{A}} y_{ij} \le 1 && \forall j \in \mathcal{G}\setminus\mathcal{H} \\ & y_{ij} \ge 0 && \forall j \in \mathcal{G}\setminus\mathcal{H}, \forall i \in \mathcal{A} \\ & \sigma : \mathcal{A} \to \mathcal{H} \text{ is a matching.} && \end{aligned} \qquad \text{(Mixed+matching)}$$

(Mixed+matching) differs from (Mixed relaxation) in two respects. Firstly, the utility is evaluated separately on $\mathcal{H}$ and $\mathcal{G}\setminus\mathcal{H}$ in the objective. Secondly, $\mathcal{H}$ is allocated to the agents by a matching. This is not a relaxation of (NSW) as the optimal integer solution may allocate multiple items in $\mathcal{H}$ to the same agent. The effect of both these changes is limited.

Theorem 4.2. *Let $\mathcal{H} \subseteq \mathcal{G}$ with $|\mathcal{H}| = |\mathcal{A}|$. If (y,σ) is an α-approximate solution of* (Mixed+matching) *then (y,σ) is a $2\alpha\gamma$-approximate solution of* (Mixed relaxation).

4.3 **Phase III**: Approximating the mixed matching relaxation

To approximate (Mixed+matching) we first remove $\mathcal{H}$. Consider the "naïve" relaxation restricted to $\mathcal{G}\setminus\mathcal{H}$, and taking the logarithm of the objective. This is the classical Eisenberg–Gale convex program that computes an equilibrium in Fisher markets with divisible items for homogeneous concave valuations [Eisenberg 1961].

$$\begin{aligned} \max \quad & \sum_{i\in\mathcal{A}} w_i \log(v_i(y_i)) && \\ \text{s.t.:} \quad & \sum_{i\in\mathcal{A}'} y_{ij} \le 1 && \forall j \in \mathcal{G}\setminus\mathcal{H} \\ & y \ge 0. && \end{aligned} \qquad \text{(EG)}$$

Given an agent-wise α-approximate solution of (EG) we can find an 2α-approximate solution to (Mixed+matching) by optimally reassigning $\mathcal{H}$.

Theorem 4.3. *Let y^* be an optimal and y be a feasible solution for* (EG) *such that $v_i(y_i) \ge \frac{1}{\alpha} v_i(y_i^*)$ for all $i \in \mathcal{A}$. Let π be a maximum weight assignment in the bipartite graph with parts $\mathcal{A}$ and $\mathcal{H}$, and edge weights $\omega_{ij} = w_i \log\left(v_i(y_i) + v_{ij}\right)$ for $i \in \mathcal{A}$, $j \in \mathcal{H}$. Then, (y,π) is 2α-approximate solution to* (Mixed+matching).

4.4 **Phase IV**: A sparse approximate solution for the mixed matching relaxation

Assuming the agents have Rado valuations, we can find an approximate solution of (Mixed+matching) with a strong sparsity property.

[2]Interestingly, in the case of symmetric agents with additive valuations the set $\mathcal{H}$ contains all items with price at least one in a spending restricted equilibrium as in [Cole and Gkatzelis 2018].

THEOREM 4.4. *Suppose that v_i are Rado valuations. We can find a 4-approximate feasible solution (y, π) to* (Mixed+matching) *with $|\text{supp}(y)| \leq 2n + m$.*

For Rado valuations, we first prove that an optimal solution of (EG) can be found in polynomial time: using (Rado ext) we show that (EG) is a rational convex program, and use the variant of the ellipsoid method for rational polyhedra. Second, we show that $|\text{supp}(y^*)| \leq n + 2m$ for any basic optimal solution y^* of (EG). Finally, we show that any such solution can be further sparsified to obtain y such that $|\text{supp}(y)| \leq 2n + m$, and in the process we lose at most the half of the value for each agent. Theorem 4.4 then follows from Theorem 4.3.

4.5 **Phase V**: Rounding the mixed integer solution

We start with a mixed integer solution (y, π) as in Theorem 4.4. We round (y, π) to (y^r, π) obtained as follows. For each $j \in \mathcal{G} \setminus \mathcal{H}$, we pick an arbitrary agent $i \in \mathcal{A}$ such that $y_{ij} > 0$ and assign j to i. By the bound on $|\text{supp}(y)|$, this amounts to setting $\leq 2n$ values y_{ij} to 0. By combining the matching π in (y, π), and the initial matching τ from **Phase I** we obtain ρ in the following lemma.

LEMMA 4.5. *Let $\mathcal{H} = \tau(\mathcal{A})$ and (y^r, π) be as above. Then we can find a matching $\rho : \mathcal{A} \to \mathcal{H}$ such that (y^r, ρ) is an $128\gamma^2$-approximate solution to* (Mixed+matching).

By Theorem 4.2 and Lemma 4.5 we obtain Theorem 2.2; the result for additive valuations requires a stronger bound in each of the above theorems and lemmas.

5. CONCLUSION AND OPEN PROBLEMS

We gave a constant-factor approximation algorithm for the asymmetric NSW problem with Rado valuations, assuming that γ is a constant. The algorithm is based on a mixed integer programming relaxation, and decomposes into a number of phases. Most reduction steps are applicable for more general settings. We only require Rado valuations for **Phase IV**, to obtain a solution with a small support.

Case in point, Li and Vondrák [2021a] very recently obtained a 380-approximation algorithm for the symmetric NSW problem under submodular valuations. This is obtained by strengthening and extending our approach, with important new techniques in **Phases III** and **IV** to deal with multilinear extension of submodular valuations. Even though submodular valuation functions do not have a concave extension, in **Phase III** they use an iterated continuous greedy method to approximately solve the non-convex program using the multilinear extension.

This settles the constant-factor approximability of the symmetric problem as an $O(n^{1-\varepsilon})$ approximation for the problem under subadditive valuations would require an exponential number of oracle queries for any fixed $\varepsilon > 0$ [Barman et al. 2020].

The constant-factor approximability of the asymmetric NSW problem remains open even for additive valuations. We note that in our approach, for additive valuations, the factor γ only appears in the reduction in **Phase II**, where we restrict each agent to receiving only a single item from the set $\mathcal{H}$.

Our work also highlights Rado valuations as an interesting class of gross substitutes valuations; this could be relevant also for other problems in mechanism design: it is a broad class including most common examples such as weighted matroid rank functions and OXS valuations, yet it has a rich combinatorial structure that can be exploited for algorithm design.

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