Exchange Markets: Proportional Response Dynamics and Beyond

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The exchange market is a basic model of an economy, where agents bring resources that they own to the market in order to exchange them for other goods that they need. There is a rich literature on the equilibrium properties of such markets starting with the work of Arrow and Debreu. In this note we survey recent results on proportional response dynamics in exchange markets with linear utilities and suggest several directions for future work.

Categories and Subject Descriptors: J.4 [**Social and Behavioral Science**]: Economics

General Terms: Algorithms, Design, Economics

Additional Key Words and Phrases: proportional response dynamics, dynamical systems, market equilibrium, exchange markets, Fisher markets, production markets

1. INTRODUCTION

The competitive (aka market) equilibrium is a key economic concept that models the allocation of resources at the steady state of an economy, where supply equals demand. The theory of general equilibrium started from the studies of Walras [Walras 1874; Brainard and Scarf 2000] and was made mathematically rigorous by [Arrow and Debreu 1954], who proved the existence of a competitive equilibrium under mild conditions.

There is a rich literature on markets, with efficient algorithms for computing competitive equilibria [Scarf 1977; Jain et al. 2003; Deng et al. 2003; Codenotti et al. 2005; Codenotti et al. 2005; Garg and Kapoor 2006; Jain 2007; Ye 2008; Devanur et al. 2008; Devanur and Kannan 2008; Garg et al. 2015; Duan and Mehlhorn 2015; Duan et al. 2016; Garg and Végh 2019; Bei et al. 2019] and computational hardness results for several markets with non-linear utilities [Nisan et al. 2007; Codenotti et al. 2006; Chen et al. 2009; Vazirani and Yannakakis 2011].

The notion of a competitive equilibrium abstracts away how equilibria are reached, if at all; this motivates dynamic processes where the agents continuously adapt their strategies to the current state of the market. On this topic, [Fisher 1983] writes:

> *Whether or not the actual economy is stable, we largely lack a convincing theory of why that should be so. Lacking such a theory, we do not have an adequate theory of value, and there is an important lacuna in the center of microeconomic theory... To only look at situations where the Invisible Hand has finished its work cannot lead to a real understanding of how that work is accomplished.*

Market Dynamics. A stream of literature studied dynamics in markets, starting with a fundamental process known as *tâtonnement*, which is a trial and error method

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of adjusting the prices: if at any point in time there is an excess demand for a good, then its price increases; if there is an excess supply instead, then the price of the good decreases. The analysis of convergence of tâtonnement dates back to [Arrow et al. 1959], while [Scarf 1960] and [Gale 1963] gave examples of cycling. For a survey, see the book of [Mas-Collel et al. 1995]. For markets with weak gross substitutes utilities, a polynomial time convergence of a discrete time process was shown by [Codenotti et al. 2005]. [Cole and Fleischer 2008] showed fast convergence for both static and ongoing markets. A similar analysis was done for markets with complementarities [Cheung et al. 2012; Cheung and Cole 2014; Avigdor-Elgrabli et al. 2014] and for Eisenberg-Gale markets [Cheung et al. 2019].

Dynamics in Networked Markets. In networked markets the agents are connected on an underlying graph and take decisions influenced by interactions with their neighbors. Fisher, exchange, and production markets with linear utilities are examples of networked markets.

Dynamics in networked markets exhibit a rich landscape of mechanisms, behaviors of the agents and patterns such as convergence or oscillations of prices, growth, and inequality [Wu and Zhang 2007; Zhang 2011; Birnbaum et al. 2011; Brânzei et al. 2018; Cheung et al. 2018; Cheung et al. 2019; Cheung et al. 2021].

We survey the proportional response dynamic in exchange markets with linear utilities from [Brânzei et al. 2021] and suggest several directions for future work.

2. EXCHANGE MARKETS

Let $[n] = \{1, \ldots, n\}$ be a set of agents, each of which brings to the market a bundle of divisible goods. We focus on the basic setting where each agent i brings an eponymous good i; w.l.o.g. there is one unit of the good.

The agents have linear utilities, described by a matrix $\mathbf{A} = (a_{i,j})_{i,j=1}^{n}$ such that $a_{i,j} \geq 0$ is the value of agent i for one unit of good j. The utility of player i for a bundle $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,n})$ is $u_i(\mathbf{x}_i) = \sum_{j=1}^{n} a_{i,j} \cdot x_{i,j}$, where $x_{i,j}$ is the amount of good j received by agent i. W.l.o.g., for each agent i there is a good j with $a_{i,j} > 0$ and for each good j, there is an agent i with $a_{i,j} > 0$.

2.1 Market Equilibria

The [Arrow and Debreu 1954] theorem states there is a *market equilibrium* $(\mathbf{p}, \mathbf{x})$, such that $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is an allocation and $\mathbf{p} = (p_1, \ldots, p_n) \neq \mathbf{0}$ a price vector where p_i is the price of good i. At these prices, each player i

(1) sells good i and collects revenue p_i, which becomes its budget, and

(2) purchases an optimal bundle $\mathbf{x}_i$ given its budget constraint.

The theorem guarantees that in the equilibrium the price of bundle $\mathbf{x}_i$ is p_i.

There are several polynomial time algorithms for computing market equilibria in exchange markets with linear utilities (see, e.g. [Jain 2007]). A strongly polynomial time algorithm was designed in [Garg and Végh 2019].

2.2 Shapley-Shubik Market Game

[Shapley and Shubik 1976] proposed a game to explain the formation of prices in markets, which is known as Trading Posts or the Shapley-Shubik game.

In our market setting, the game proceeds as follows. Each player i brings to the market one unit of good i and a budget B_i, which has no intrinsic value but is used to facilitate exchange. Then player i submits bids $b_{i,j} \geq 0$, where $\sum_{j=1}^{n} b_{i,j} = B_i$.

Each good j is allocated in proportion to the bid amounts, so player i receives

$$x_{i,j} = \begin{cases} \frac{b_{i,j}}{\sum_{k=1}^{n} b_{k,j}}, & \text{if } b_{i,j} > 0 \\ 0, & \text{otherwise} \end{cases}$$

The sum of bids $p_j = \sum_{k=1}^{n} b_{k,j}$ at item j is the price of the good.

The existence and quality of equilibria of the Shapley-Shubik game have been studied under various names such as Trading posts and proportional share mechanism (see, e.g., [Feldman et al. 2009]).

2.3 Proportional Response Dynamics

Tâtonnement does not converge to market equilibria in exchange markets when the players have linear utilities, thus different explanations for how market equilibria may be reached are needed.

We describe a dynamic where the players repeatedly bring to the market one unit of their good and a budget. The players submit bids and receive allocations according to the Shapley-Shubik game. Then the seller of each good collects the money made from selling, which becomes its budget in the next round, and updates their bids in proportion to the contribution of each good in its utility.

This dynamic is known as proportional response, and was studied in exchange markets in [Brânzei et al. 2021].

DEFINITION 2.1. (Proportional Response) *At each time $t = 0, 1, \ldots$, each player i brings one unit of an eponymous good and the next steps take place:*

`Shapley-Shubik Game:` *Each player i bids $b_{i,j}(t)$ on every good j. Player i gets from each good j a fraction proportional to their bid:*

$$x_{i,j}(t) = \begin{cases} \frac{b_{i,j}(t)}{\sum_{k=1}^{n} b_{k,j}(t)}, & \textit{if } b_{i,j}(t) > 0 \\ 0, & \textit{otherwise} \end{cases}$$

The player's utility is $u_i(t) = \sum_{k=1}^{n} a_{i,k} \cdot x_{i,k}(t)$.

`Bid Update:` *Player i collects the money from selling, which becomes its budget in the next round: $B_i(t+1) = \sum_{k=1}^{n} b_{k,i}(t)$ and updates the bids proportionally to the contribution of each good in their utility :*

$$b_{i,j}(t+1) = \left(\frac{a_{i,j} \cdot x_{i,j}(t)}{u_i(t)}\right) \cdot B_i(t+1)$$

2.4 Example

Consider the two-player market in Figure 1. The valuations are $\mathbf{a}_1 = (2, 5)$ and $\mathbf{a}_2 = (3, 4)$; the bid amounts are shown as coins on the edges.

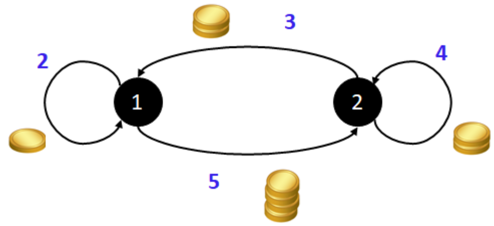


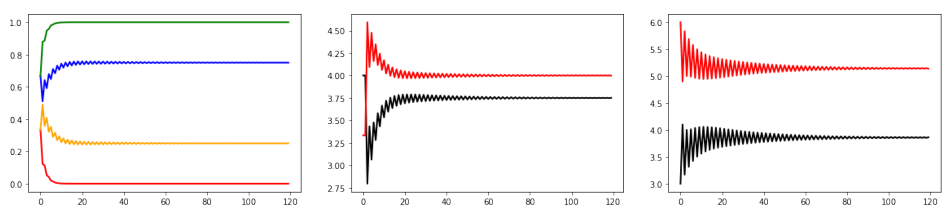
Fig. 1: A two player market with valuations and bids shown as coins on the edges. For example, player 1 has value 2 for one unit of good 1 and bids one dollar on good 1; it has value 5 for one unit of good 2 and bids four dollars on it.

Let $\mathbf{b}_i(t) = (b_{i,1}(t), \ldots, b_{i,n}(t))$ denote the bids submitted by player i in round t, where $b_{i,j}(t)$ is the bid of player i for good j at time t. The first round of the dynamic proceeds as follows:

Shapley-Shubik Game: Players submit bids $\mathbf{b}_1(0) = (1, 4)$ and $\mathbf{b}_2(0) = (2, 2)$. The prices induced by these bids are $p_1(0) = b_{1,1}(0) + b_{2,1}(0) = 1 + 2 = 3$ and $p_2(0) = b_{1,2}(0) + b_{2,2}(0) = 2 + 4 = 6$.
The allocation is $x_{i,j}(0) = \frac{b_{i,j}(0)}{p_j(0)}$: $\mathbf{x}_1(0) = (\frac{1}{3}, \frac{4}{6})$ and $\mathbf{x}_2(0) = (\frac{2}{3}, \frac{2}{6})$. The utilities are $u_1(0) = 2 \cdot \frac{1}{3} + 5 \cdot \frac{4}{6} = 4$ and $u_2(0) = 3 \cdot \frac{2}{3} + 4 \cdot \frac{2}{6} = 10/3$.

Bid Update: The updated budgets are $B_1(1) = p_1(0) = 3$ and $B_2(1) = p_2(0) = 6$. The updated bids are $b_{i,j}(1) = \frac{a_{i,j} \cdot x_{i,j}(0)}{u_i(0)} \cdot B_i(1)$. Thus player 1's bids are $b_{1,1}(1) = \left(\frac{2 \cdot 1/3}{4}\right) \cdot 3 = 1/2$ and $b_{1,2}(1) = \left(\frac{5 \cdot 4/6}{4}\right) \cdot 3 = 5/2$. A similar calculation shows player 2's bids are $b_{2,1}(1) = 36/10$ and $b_{2,2}(1) = 24/10$.

Figure 2 shows a simulation of this dynamical system for 120 rounds.



(a) Allocations (b) Utilities (c) Prices

Fig. 2: A simulation of proportional response dynamics on the two-player market from Figure 1. The X axis shows the time step, which ranges from 0 to 120. Figure (a) shows the fraction received by player 1 from good 1 in red and from good 2 in blue; the fraction received by player 2 from good 1 is shown in green and from good 2 in orange. Figure (b) shows the utility of player 1 in red and of player 2 in black. Figure (c) shows the price of good 1 in red and of good 2 in black.

2.5 Properties of Proportional Response in Exchange Markets

We are interested in the long term behavior of the dynamical system in Definition 2.1. [Brânzei et al. 2021] show the dynamic converges to market equilibrium allocations and utilities for any non-degenerate initial bids (i.e. where $b_{i,j}(0) > 0$ whenever $a_{i,j} > 0$).

Potential function. The key to analyzing the dynamical system is to find a potential function. Let $\mathbf{p}^* = (p_1^*, \ldots, p_n^*)$ and $\mathbf{x}^* = (x_{i,j}^*)_{i,j=1}^n$ be an equilibrium price vector and allocation, respectively. Let $b_{i,j}^* = p_j^* \cdot x_{i,j}^*$ for each $i, j \in [n]$. W.l.o.g., the total amount of money in the economy is 1: $\sum_{i,j=1}^n b_{i,j}(0) = 1$.

For each $t \in \mathbb{N}$ and $i, j \in [n]$, let

$$z_{i,j}(t) = \begin{cases} \left(\frac{b_{i,j}^*}{b_{i,j}(t)}\right)^{b_{i,j}^*} & if\ b_{i,j}^* > 0\ and\ b_{i,j}(t) > 0 \\ 1 & otherwise \end{cases} \tag{1}$$

Define the function

$$f(t) = \prod_{i,j=1}^n z_{i,j}(t) \tag{2}$$

Then $f(t)$ is a potential function, with $f(t+1) < f(t)$ at all times t where the utilities are not the equilibrium utilities. By using connections between the exchange market and the Eisenberg-Gale convex program, it can be shown that the fixed point utilities coincide with the market equilibrium utilities. This will also imply that the allocations converge to market equilibrium allocations.

Bid cycling. While the allocations and utilities converge, the bids and prices may cycle. For example, consider the economy and initial bids in Figure 3. Players 1 and 2 will continue swapping their budgets throughout time, so the prices of the two goods will oscillate.

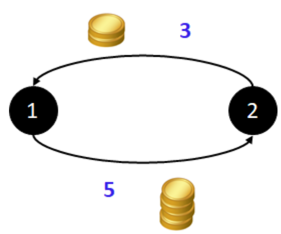


Fig. 3: Two player market where prices cycle in the proportional response dynamics. The valuations are $a_{1,1} = 0$, $a_{1,2} = 5$, $a_{2,1} = 3$, $a_{2,2} = 0$. The initial bids are $b_{1,1}(0) = 0$, $b_{1,2}(0) = 4$, $b_{2,1}(0) = 2$, and $b_{2,2}(0) = 0$. The price of good 1 will alternate between 2 and 4 throughout time.

The limit cycles of the dynamic can be characterized in the following way: there exist equivalence classes of players such that

(i) within each class, the ratio of price to equilibrium price is a constant, and

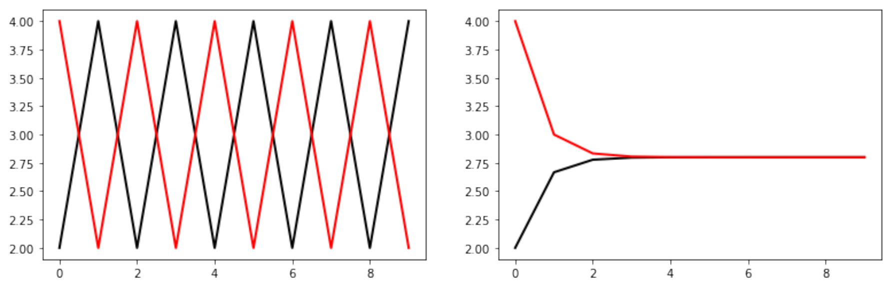
(ii) the classes form a cycle, where the players in each class only buy goods from the players in the next class in the cycle.

Lazy dynamic. [Brânzei et al. 2021] also define a lazy version of the dynamic, where the players may save money for later; they show this converges in everything: utilities, allocations, and prices.

The definition of the lazy dynamic is similar to Definition 2.1, except in each round t, each player i only spends a fraction α_i of its budget and saves the rest in the bank. After the bids and allocations are computed, player i gathers all their money, both from selling their good and the money saved in the bank. This amount is the player's total budget, which is again split in a fraction of α_i used for bidding in round $t+1$ and a fraction of $1-\alpha_i$ saved in the bank.

The cycling of the prices appears in the extreme case of the lazy dynamic where each player spends their whole budget in each round, i.e. $\alpha_i = 1$ for each player i.

The difference between the lazy and non-lazy dynamic is illustrated in Figure 4.



(a) (b)

Fig. 4: Prices for the market in Figure 3, showing the difference between the non-lazy and lazy dynamic. Figure (a) shows the dynamic in Definition 2.1, where the players spend their whole budget in each round. Figure (b) shows the lazy version of the dynamic, where player 1 spends a fraction $\alpha_1 = 1/2$ of their money in each round and player 2 spends a fraction of $\alpha_2 = 2/3$.

2.6 History of proportional response dynamics

Fisher markets. Proportional response dynamics were first studied in Fisher markets as an alternative to tâtonnement, since the latter process does not converge to market equilibria. [Zhang 2011] showed that proportional response converges to market equilibria in Fisher markets with linear utilities. Fisher markets are a special case of exchange markets, where the graph is bipartite and the agents are divided in buyers and sellers. Each seller brings a good for sale and each buyer brings a budget. The buyers only have value for the goods while the sellers only have value for the money.

[Birnbaum et al. 2011] further showed the process is equivalent to gradient descent on a convex program that captures the equilibria for linear utilities. [Cheung et al. 2018] show that proportional response dynamics converges to market equilibria for the entire range of CES utilities including *complements*, with linear utilities on one extreme and *Leontief* utilities on the other extreme. [Cheung et al. 2019] show that the dynamics stays close to equilibrium even when the market parameters are changing slowly over time, once again for CES utilities.

Production markets. [Brânzei et al. 2018] studied proportional response dynamics in production markets, where the dynamic leads to *growth* of the market, i.e. the amount of goods produced increase over time, but also to growing *inequality* between the players on the most efficient production cycles and the rest. In particular, the dynamic learns through local interactions a global feature of the exchange graph—the cycle with the highest geometric mean.

There is a similarity between proportional response dynamics in the production market and the Lotka-Volterra model [Lotka 1910; Volterra 1928; Wangersky 1978] which studies interdependence of animals and how they help or destroy each other based on their interactions and reproduction.

Tit-for-tat in exchange markets. [Wu and Zhang 2007] studied a tit-for-tat dynamic, which does not use money, in a special type of exchange market where the goods have common value (i.e. the value of any good j is the same for every player i: $v_{i,j} = v_j$), and showed it converges to market equilibria. This setting is relevant to networking applications.

Connections with blockchain mining. There are connections between proportional response dynamics, and blockchain mining, since miners in the blockchain setting succeed with probability roughly proportional to the effort invested. [Cheung et al. 2021] studied learning dynamics in several production economies, such as blockchain mining, peer-to-peer file sharing and crowdsourcing. They also study Fisher markets and show that proportional response dynamics converges to market equilibria when the players have quasi-linear utilities.

3. DISCUSSION AND FUTURE DIRECTIONS

Several directions remain open. An immediate question is the following: What is a suitable generalization of the dynamic for exchange markets in the most general case where each player brings multiple goods?

What are natural generalizations of proportional response dynamics in networked markets, such as Fisher and exchange markets with linear utilities or production markets with linear production? For example, consider the process where the allocation is done via the Shapley-Shubik mechanism, but the players change their investment fractions using multiplicative weight updates. Does this process converge to a market equilibrium, in either Fisher or exchange markets with linear utilities? What other allocation mechanisms (e.g. auctions) can be used for allocating goods given that players in networked markets will repeatedly interact with their neighbors? What features of such systems are relevant for finding potential functions?

It would also be interesting to explore further the connections between proportional response dynamics and blockchain mining [Cheung et al. 2021]. How should resource allocation mechanisms be designed given the inherent dynamic nature of the blockchain system?

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