Convergence, Steady State, and Global Gains of Agent-Based Coalition Formation in E-markets

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In this paper, we present a rigorous analysis of an agent-based coalition formation mechanism in e-markets proposed by Lerman and Shehory. While the agent-based coalition formation shows good performance through simulations, our analysis provides guidelines for agent designers to simplify the system design. We show that the coalition formations with different initial distributions converge to a unique steady state. The steady state, which represents both the final coalition distribution and the global utility gain, is proven to be determined by buyer agents' local strategies. The global utility gain is shown to increase as the number of buyer agents increases. In a system of uniform-attachment-uniform-detachment rates, the global utility gain is proven to increase as the detachment rate decreases.

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1. INTRODUCTION

In both e-commerce and conventional commerce, participants in an/a (e-)market can benefit by forming coalitions (e.g. [1-4]). While buyers can obtain discounts by purchasing goods in a group, sellers can benefit by selling more units of the goods. In a seller' site, the seller specifies both the fixed prices of the goods and the discount rules for different sizes of coalitions. [5-7] proposed a mechanism for coalition formation of agents in an e-market using minimal computation and communication costs. In [5-7], each buyer is represented by an agent and the autonomous agent wanders in the e-market searching for large coalitions without complete information

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about the e-market. To determine parameters that are essential in the coalition formation and the effects they have on the global utility gain, Lerman and Shehory [5] formulated a model to describe the coalition formation procedure.

Although [5-7] presents a novel approach for coalition formation, only empirical results were obtained. The contributions of this work are: (1) proving the convergence of the coalition formations; (2) giving the explicit connection between agents' strategies and the final steady state; (3) showing the relationship between local agents' strategies and the global utility gain.

In this paper, we present a rigorous theoretical analysis for the proposed coalition formation mechanism in e-markets. We prove that the coalition formations with different initial distributions converge to a unique steady state when the maximum size of a coalition is two. For coalitions with sizes greater than two, we show the convergence by numerical simulations. The steady state, which determines both the final coalition distribution and the global utility gain, is expressed by agents' simple local strategies. The global utility gain is shown to increase as the amount of buyer agents increases. In a specific multiple agent system of uniform-attachment-uniform-detachment rates, the global utility gain is proven to increase as the attachment rate increases or as the detachment rate decreases.

2. AGENT-BASED COALITION FORMATION

Consider a system of many unaffiliated buyer agents in an e-market and each agent has incomplete information about the e-market (e.g. an agent does not know the coalition size at each seller's site before visiting it). To search for large coalitions, each buyer agent moves among sellers' sites, collects information about the coalitions, and decides on joining or leaving an encountered coalition. Due to manufacturing and distribution constraints as pointed out in [5], a seller agent can specify a maximum size of coalitions, which is denoted by m. Then a buyer agent can only join the coalitions whose sizes are smaller than m. Agents can leave or join the coalition in a seller's site, or form a new coalition if there is no other buyer agents there. After joining a coalition, a buyer agent can also leave the coalition and search for larger coalitions. Buyer agents' actions are joining and leaving. Strategies for the two actions will determine the group behavior.

In [5], some assumptions were given to capture the characteristics of the proposed coalition formation:

(1) Sellers are homogeneous in the sense that they offer the same price and the same discount rates.

(2) Buyer agents are homogeneous in the sense that they buy the same goods and have the same strategies of leaving and joining.

(3) Initially all the buyer agents are unaffiliated.

(4) Buyer agents move on the e-market and meet sellers and other buyer agents randomly.

(5) Large coalitions are beneficial for buyer agents to obtain more discounts.

(6) Each buyer agent can independently decide on leaving or joining an encountered coalition, but it will not join a coalition with size m - the maximum size.

(7) The total number of buyer agents in the system remains constant, which is denoted by M.

The procedure of coalition formation mechanism is outlined below, where buyer agents require minimal computations and communication costs and need not have global knowledge about coalitions on the e-market.

Procedure 1

```
Initially all the buyer agents are unaffiliated;
START: a buyer agent moves on the e-market
    and meets a seller of coalition size n;
    if n = m or the buyer agent leaves the coalition,
        goto START;
    endif
    if the buyer agent joins the coalition,
        the size of the coalition becomes n + 1;
    endif
    if the buyer agent leaves the coalition of size n + 1,
        the size of the coalition becomes n;
        goto START;
    endif
EXIT: the buyer agent joins the coalition and doesn't leave.
```

3. MATHEMATICAL CHARACTERIZATION

Procedure 1 is characterized mathematically in this section. We focus on two quantities: the number and size of coalitions.

Let $r_1(t)$ be the number of unaffiliated agents at time t. For $2 \leq n \leq m$, let $r_n(t)$ be the number of coalitions of size n at time t. The coalition formation mechanism we are concern about can be quantitatively described as follows [5]:

$$r_1^{'} = 2L_2 r_2 - 2J_1 r_1^2 + \sum_{n=3}^{m} L_n r_n - r_1 \sum_{n=2}^{m-1} J_n r_n,$$

$$\vdots$$

$$r_n^{'} = -L_n r_n + J_{n-1} r_1 r_{n-1} - J_n r_1 r_n + L_{n+1} r_{n+1}, 2 \leq n \leq m-1, \qquad (1)$$

$$\vdots$$

$$r_m^{'} = -L_m r_m + J_{m-1} r_1 r_{m-1},$$

where $J_n > 0, L_n > 0, r_n \geq 0, \sum_{n=1}^{m} nr_n(0) = M$.

Here the term $r_n^{'}$ denotes $dr_n(t)/dt$, which is the rate of change in the number of coalitions of size n, $1 \leq n \leq m$. Parameter J_n denotes the rate at which unaffiliated agents encounter and join coalitions of size n. Parameter L_n denotes the rate at which an agent leaves coalitions of size n. As we have discussed in Section 2, parameters J_n and L_n come from agents' experiences after they have wandered and queried sellers that they have visited in the e-market for a period of time.

Agents' actions in coalition formations are accurately described in system of equations (1). In the first equation of (1), the term "$2L_2r_2$" specifies that one coalition of size 2 becomes two unaffiliated agents after an agent leaves the coalition. The term "$-2J_1r_1^2$" specifies that two unaffiliated agents become one coalition of size 2 after an agent joins the coalition.

For $3 \leq n \leq m$, the term "L_nr_n" specifies that one coalition of size n becomes one unaffiliated agent and one coalition of size $n-1$ after an agent leaves the coalition.

For $2 \leq n \leq m-1$, the term "$-J_nr_1r_n$" specifies that one unaffiliated agent and one coalition of size n become one coalition of size $n+1$ after an agent joins the coalition. Inductively, each term in the other equations of (1) can also be explained in the similar fashion.

By (1), we have $\sum_{n=1}^{m} nr_n^{'} = 0$, then $\sum_{n=1}^{m} nr_n(t) = \sum_{n=1}^{m} nr_n(0) = M$ as $t > 0$, which is consistent with the seventh assumption in Section 2.

4. MAIN RESULTS

In this section, we show that the coalition formation proposed in [5] converges to a unique steady state (equilibrium) (section 4.1) and prove that the global utility gain increases with the number of buyer agents (section 4.2). For the case of uniform-attachment-uniform-detachment rates, we prove that the global utility gain increases as the attachment rate increases or as the detachment rate decreases (section 4.2).

4.1 Convergence

Here we prove both the existence and uniqueness of a steady state (equilibrium) for the coalition formation proposed in [1] and express it with local agents' strategies (Theorem 4.1). Furthermore, we show that for a coalition size not exceeding two, the system converges to the steady state (Theorem 4.2).

THEOREM 4.1. *There exists a unique equilibrium (steady state) of (1).*

Proof. Let $r^* = (r_1^*, r_2^*, ..., r_m^*)$ be the equilibrium of (1). It follows from the definition of equilibrium that $\frac{dr_n}{dt}|_{r^*} = 0, \quad 1 \leq n \leq m$, then we have

$$2L_2r_2 - 2J_1r_1^2 + \sum_{n=3}^{m} L_nr_n - r_1 \sum_{n=2}^{m-1} J_nr_n = 0,$$

$$\vdots$$

$$-L_nr_n + J_{n-1}r_1r_{n-1} - J_nr_1r_n + L_{n+1}r_{n+1} = 0, 2 \leq n \leq m-1, \qquad (1^{'})$$

$$\vdots$$

$$-L_{m-1}r_{m-1} + J_{m-2}r_1r_{m-2} - J_{m-1}r_1r_{m-1} + L_mr_m = 0,$$

$$-L_mr_m + J_{m-1}r_1r_{m-1} = 0.$$

It follows from the m-th equation of $(1^{'})$ that

$$r_m^* = \frac{J_{m-1}}{L_m} r_1^* r_{m-1}^*.$$

By the $(m-1)$-th and the m-th equations of $(1^{'})$, we have

$$r_{m-1}^* = \frac{J_{m-2}}{L_{m-1}} r_1^* r_{m-2}^*.$$

Using the method inductively, we have:

$$r_n^* = \frac{J_{n-1}}{L_n} r_1^* r_{n-1}^*, 2 \leq n \leq m.$$

Therefore

$$r_2^* = \frac{J_1}{L_2} r_1^{*2},$$

$$r_3^* = \frac{J_2}{L_3} r_1^* r_2^* = \frac{J_1 J_2}{L_2 L_3} r_1^{*3}.$$

Then we have:

$$r_n^* = \frac{J_1 J_2 ... J_{n-1}}{L_2 L_3 ... L_n} r_1^{*n}, 2 \leq n \leq m,$$

that is, $r_n^* = q_n r_1^{*n},\ 1 \leq n \leq m$, where

$$q_1 = 1, q_n = \frac{J_1 J_2 ... J_{n-1}}{L_2 L_3 ... L_n}, 2 \leq n \leq m.$$

Since $\sum_{n=1}^{m} n r_n^* = \sum_{n=1}^{m} n q_n r_1^{*n} = M$, let $G(y) = \sum_{n=1}^{m} n q_n y^n - M$, then r_1^* is a solution of $G(y) = 0$, that is, $G(r_1^*) = 0$. By the definition of $G(y)$, we have

$$G(0) = -M < 0,\ G(M) > 0.$$

By computing the differential value $dG(y)/dy$, we have

$$G^{'}(y) = \sum_{n=1}^{m} n^2 q_n y^{n-1} > 0 \ \text{ as } y > 0,$$

then the function $G = G(y)$ is monotonous. Hence, there exists a unique solution of $G(y) = 0$ in $(0, M)$. That is, there is a unique solution of r_1^*. It follows from $G^{'}(r_1^*) > 0$ and the *Implicit Function Theorem* [8] that r_1^* can be a function of q_n, that is, a function of strategies L_n and J_n. Since $r_n^*(2 \leq n \leq M)$ can be expressed by r_1^*, then there is a unique equilibrium of (1) and the equilibrium can be expressed as a function (vector) of strategies L_n and J_n. Q.E.D.

Theorem 4.1 shows both the existence and uniqueness of a steady state (equilibrium) in the coalition formations of (1) and the steady state can be expressed by local agents' strategies of leaving and joining a coalition. In the steady state, the number of coalitions remains unchanged.

Results from Theorem 4.1 establishes a relationship between agents' local strategies (the decision to leave their coalitions and to join other coalitions) and the distribution of coalitions at the steady state. This sets the stage for later discussion of utility gains in the next section.

Theorem 4.2 shows that for coalition size not exceeding two, the system converges to the steady state.

THEOREM 4.2. *In case $m = 2$, all the solutions of (1) converge to the unique equilibrium.*

Proof. Let $m = 2$, then the system (1) becomes

$$r_1^{'} = 2L_2r_2 - 2J_1r_1^2,$$

$$r_2^{'} = -L_2r_2 + J_1r_1^2, \tag{2}$$

where $r_1 + 2r_2 = M$. Since $r_2 = \frac{1}{2}(M - r_1)$, then the first equation of (2) becomes:

$$r_1^{'} = F(r_1),$$

where $F(r_1) = L_2M - L_2r_1 - 2J_1r_1^2,\ 0 \leq r_1 \leq M.$

For the interval $[0, M]$, since

$$r_1^{'}(0) = F(0) = L_2M > 0,$$

$$r_1^{'}(M) = F(M) = 2J_1M^2 < 0,$$

then $r_1^{'} > 0$ at $r_1 = 0$, hence r_1 increases with time. Similarly, $r_1^{'} < 0$ at $r_1 = M$, then r_1 decreases with time. Since there is a unique equilibrium r_1^* of (2) on $[0, M]$ (see Theorem 4.1), then all the solutions $r_1(t)$ of (2) initiated from $[0, M]$, converge to the equilibrium r_1^*. Since $r_2 = \frac{1}{2}(M - r_1)$, then all the solutions of (2) converge to the unique equilibrium $(r_1^*, \frac{1}{2}(M - r_1^*))$. Q.E.D.

Theorem 4.2 shows that even though there is no central control, the process of coalition formation among autonomous and self-interested agents can still reach a steady state in which the distribution of coalitions remains unchanged. It shows that undesirable system's behaviors such as periodic oscillations and chaos do not occur.

For $m \geq 3$, the unique equilibrium is shown to be globally asymptotically stable through many numerical simulations. However, due to space limitation only a special case is shown below:

Let $m = 10$,$M = 10000$, $J_s = L_{m-s} = 0.9$,$0 \leq s \leq 4$; $J_j = L_{m-j} = 0.001$,$6 \leq j \leq 9$,$J_5 = L_5 = 0.00001$. In Fig. 1(a), it can be seen that the number of coalitions of size $i(1 \leq i \leq 10)$ stabilizes with time. In Fig. 1(b), it can be seen that even with five different initial distributions of coalitions patterns, the coalitions of size 5 converge to the same formation pattern.

4.2 Global Utility Gain

The global utility gain at the equilibrium measures the total discount of the system resulting from the coalition formation. In this section, we establish the relationships

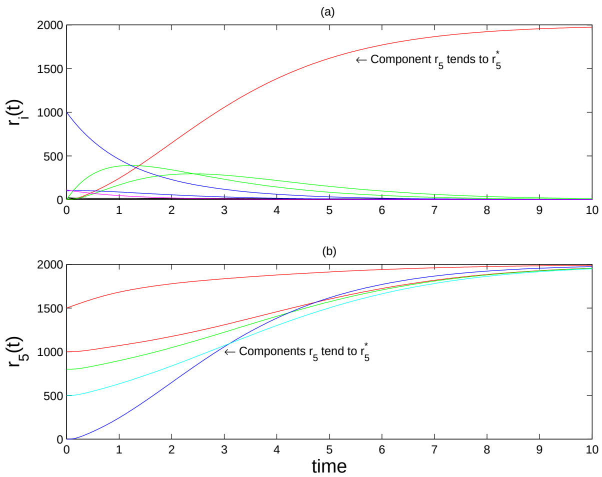

Fig. 1. Let $m = 10, M = 10000$, $J_n = L_{m-n} = 0.9, 0 \leq n \leq 4$; $J_j = L_{m-j} = 0.001, 6 \leq j \leq 9, J_5 = L_5 = 0.00001$. Coalition formation tends to the equilibrium r^*, where $r_5^* = 2000$ and other components are zero.

between (i) global utility gain and the number of agents (Theorem 4.3), and (ii) between global utility gain and local agents' strategies (Theorem 4.4).

Let $r^* = (r_1^*, r_2^*, ..., r_m^*)$ be the unique equilibrium of (1). Let $q_1 = 1, q_n = \frac{J_1 J_2 ... J_{n-1}}{L_2 L_3 ... L_n}, 2 \leq n \leq m$. The price that an unaffiliated buyer agent pays for the product is p, and the coalition price that each member pays is

$$p_n = p - \Delta p(n-1),$$

where Δp is some price decrement, and n is the coalition size. The global utility gain for all buyer agents is defined as

$$U = Mp - \sum_{n=1}^{m} p_n n r_n^*.$$

THEOREM 4.3. *The global utility gain increases as the number of agents increases, that is, $U = U(M)$ of (1) satisfies $\frac{dU}{dM} > 0$.*

Proof. By the definitions of U and p_n, we have

$$U = Mp - \sum_{n=1}^{m} [p - \Delta p(n-1)] n r_n^*$$

$$= Mp - (p + \Delta p) \sum_{n=1}^{m} n r_n^* + \Delta p \sum_{n=1}^{m} n^2 r_n^*$$

Since $r_n^* = q_n r_1^{*n}$ and $\sum_{n=1}^{m} n r_n^* = \sum_{n=1}^{m} n q_n r_1^{*n} = M$, we have

$$U = \Delta p(-M + \sum_{n=1}^{m} n^2 q_n r_1^{*n}). \tag{5}$$

Since $\sum_{n=1}^{m} nq_n r_1^{*n} = M$, then M is a strictly monotonous function of r_1^*. It follows from the *Implicit Function Theorem* [8] that r_1^* is also a strictly monotonous function of M and

$$\frac{dr_1^*}{dM} = (\sum_{n=1}^{m} n^2 q_n r_1^{*(n-1)})^{-1} > 0.$$

Then

$$\frac{dU}{dM} = \Delta p(-1 + \sum_{i=1}^{m} i^3 q_i r_1^{*(i-1)} \frac{dr_1^*}{dM}) > 0.$$

Q.E.D.

Theorem 4.3 shows that everyone in the market (buyers and sellers) benefits further if more buyers join the coalition formation. This is because more buyers result in larger coalitions where buyers can obtain more discounts and sellers can sell larger bundles of the goods.

For example, the Campus bookstore tells students that if they come and purchase their text books in a group of 10, they would get a discount of 5% and if they come in a group of 20 they get an additional 3% discount. Using this strategy, even though the bookstore sells each text book for less, she benefits from luring more students who might otherwise purchase their books elsewhere and students get more discount if they buy in a bigger group.

Consider the case of uniform-attachment-uniform-detachment rates: $J_n = J, L_n = L, 1 \leq n \leq m$. Let $\tau = Jt$ and $a = L/J$, then system (1) becomes (we still denote τ by t):

$$r_1' = 2ar_2 + \sum_{k=3}^{m} ar_k - 2r_1^2 - r_1 \sum_{k=2}^{m-1} r_k,$$

$$\vdots$$

$$r_n' = -ar_n + ar_{n+1} + r_1 r_{n-1} - r_1 r_n, \; 2 \leq n \leq m-1,$$

$$\vdots$$

$$r_m' = -ar_m + r_1 r_{m-1}.$$

Let $x = r_1^*/a$, then the equilibrium $r^* = (r_1^*, r_2^*, ..., r_m^*)$ satisfies

$$r_n^* = ax^n, n = 1, ..., m,$$

$$\sum_{n=1}^{m} nx^n = Ma^{-1}. \tag{6}$$

By (6), a is a monotonous function of x, then it follows from the *Implicit Function Theorem* [8] that x is also a monotonous function of a and

$$\frac{dx}{da} = -\frac{M}{a^2 \sum_{n=1}^{m} n^2 x^{n-1}} < 0. \tag{7}$$

Since $x = x(a)$ is a smooth function, then $U = -M\Delta p + a\Delta p \sum_{n=1}^{m} n^2 x^n$ is a smooth function of the parameter a. Furthermore, it follows from (6) that

$$U = U(a) = -M\Delta p + mM\Delta p - a\Delta p \sum_{n=1}^{m-1} (m-n)nx^n. \tag{8}$$

THEOREM 4.4. *The global utility gain increases as the detachment rate decreases or as the attachment rate increases, that is,* $U = U(a)$ *with* $a = L/J$ *satisfies* $\frac{dU}{da} < 0$.

Proof. By (8), we obtain that $\frac{dU}{da} = -P(x)\Delta p(\sum_{n=1}^{m} n^2 x^n)^{-1}$ through the replacement of dx/da in (7), where $x = r_1^*/a$, $P(x) = \sum_{k=2}^{2m-1} C_k x^k, C_k = \sum_{n=1}^{k-1}(m - n)n(k-n)(k-2n)$. To evaluate the sign of coefficient C_k, k is considered to be an odd number and an even number respectively.

Suppose $k = 2s + 1$ is an odd number, let $j = 2s + 1 - n, s + 1 \leq n \leq 2s$, then $C_k = \sum_{n=1}^{s} n(k-n)(k-2n)^2 > 0$.

Suppose $k = 2s$ is an even number, let $j = 2s - n, s + 1 \leq n \leq 2s - 1$, then $C_k = \sum_{n=1}^{s} n(k-n)(k-2n)^2 > 0$.

Therefore $C_k > 0$ for $2 \leq k \leq 2m - 1$, that is, $dU/da < 0$. Q.E.D.

Theorem 4.4 shows that both buyers and sellers benefit if unaffiliated agents are to join coalitions rather than leaving coalitions and more coalitions are formed. This is because sellers benefit by selling more units of a product even though at a lower price while buyers benefit by paying less if they buy the goods in a larger group.

For example, 20 students want to buy their text books in some bookstores and the bookstores give the same discount rates. If the 20 students purchase their books in a bookstore without anyone changing her(his) mind (i.e., leaving the group), they would get a discount of 8%. If only ten of them stay in the group (i.e., purchase their books in the bookstore) while the other ten students leave the group and purchase their books in another bookstore, then both of the groups would get a discount of 5%. Hence, with more students leaving the group, the discount rates decrease and the benefits for both the bookstore and the students decrease.

5. CONCLUSION

Based on a characterization of agent-based coalition formations proposed by Lerman and Shehory, we show the convergence of coalition formations and construct the relationship between local agents' strategies and the global utility gain through the steady state. By proving that the coalition mechanism converges to a steady state, we show that undesirable system's effect such as periodic oscillations and chaos may not take place. Consequently, the system is stable and more predictable. From an agent designer's perspective this helps to simplify the design of the system.

Since the steady state can be expressed by agents' local strategies and the steady state represents the final coalition distribution, the final coalition distribution can be predicted from individual agents' strategies. Additionally, since the steady state also determines the global utility gain, then by expressing the steady state with individual agents' strategies, the relationship between local agents' strategies and the global utility gain is established. This relationship provides guidelines for optimizing the global utility gain by selecting the appropriate strategies. Based on expression (5) in Section 4.2, by adopting a set of strategies, the global utility gain can be determined. (5) shows that the global utility gain can be deduced from local strategies.

Our results provide some insights for designers of agent systems involving coalition formation. If no agent leaves its present coalition in search of larger coalitions, then there are only small coalitions (or unaffiliated agents) in the system and the global utility gain would be very small. This is because in smaller coalitions, buyers will get smaller discounts and sellers sell fewer units of the product. However, if too many agents prefer to leave their present coalitions by wandering in the market searching for larger coalitions, Theorem 4.4 shows that the global utility gain would be very small. Only when a small fraction of the agents (but not zero) change their coalitions (in search of large ones), while others stay in their present coalitions would the global utility gain be large. Hence, by tuning the attachment-rate and detachment-rate, agent designers can better analyze and enhance the system's performance by exploiting the relationship between agents leaving/joining coalitions and the corresponding global utility gain.

In its present form, this work does not deal with optimizing the global utility gain by selecting strategies. Our proof about the convergence is strict for the case that the maximum coalition size is two. For the case that the maximum coalition size is larger than two, the proof is still open. We do not consider maximal size $n > 2$ because it involves n differential equations and solving a system of n differential equations is difficult and to the best of our knowledge there is no general method or tool for such analysis unlike the case when $n = 2$ where only 2 differential equations are involved. An extension in applying our results to a real situation is using simulations to determine the parameters in the model.

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