Item Pricing for Revenue Maximization

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In this note we report recent results on item pricing for revenue maximization in the presence of buyers with complex, unknown preferences. We focus on two important classes of settings: buyers with general valuations for the case of items in unlimited supply, and buyers with subadditive valuations for the case of items in limited supply.

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Pricing items for sale is an important problem in Economics, and to a large extent describes today's trading practices. From a theoretical perspective, the issue of market equilibrium prices has received enormous attention over the years. In this note we report on our recent work [Balcan et al. 2008] that concentrates instead on the fundamental problem of *revenue maximization*. We consider a single seller of n goods (items) who must set prices on the items before the arrival of a sequence of customers with complex, unknown preferences (e.g., think of a store or a yard sale), and whose goal is to maximize his revenue. We prove that a simple posted single pricing scheme yields revenue guarantees that are the best guarantees known for this problem for two important classes of settings: buyers with *general* valuations for the case of items in unlimited supply, and buyers with *subadditive* valuations for the case of items in limited supply. Moreover, our results also yield truthful mechanisms with revenue guarantees for combinatorial auctions. Note that while much work on combinatorial auctions considers bundle-pricing mechanisms (such as based on VCG [Cramton et al. 2005; Nisan 2007]), the vast majority of

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transactions in today's world are conducted via pricing on items, and thus it is important to understand what guarantees are possible in such a setting.

Formally, the problem we consider is the following. A single seller has n items each in limited or unlimited supply. There are m buyers with quasi-linear utilities who arrive in an arbitrary order and who have unknown and potentially highly complex valuations over subsets of these items.[1] The seller must assign prices to the items, and then buyers arrive one at a time and purchase whatever subset of the remaining items gives them maximum utility. The goal of the seller is to maximize his total revenue. Since prices are fixed before buyers arrive, all revenue guarantees also apply trivially to the problem of designing *truthful mechanisms* with revenue guarantees in the context of combinatorial auctions. As an upper bound on the revenue that the seller can hope to extract from the buyers we use the optimum social welfare, which is the maximum possible sum of buyers' valuations in any allocation. This is the most revenue the seller could extract even if the seller could price each bundle differently for every buyer.

In the unlimited supply setting, we show that for buyers with *general* valuation functions, choosing a single price at random from an appropriate distribution to assign to all items guarantees the retailer an expected revenue within a logarithmic factor of the total social welfare. This extends work of Guruswami et al. [2005] who show this for the special cases of unit-demand and single-minded customers.[2]

In the limited supply setting, no good approximation is possible for general valuation functions, so instead we consider several important classes of valuations functions which have been studied in the social-welfare context: submodular, XOS, and more generally, subadditive valuation functions [Feige 2006; Lehmann et al. 2002; Lavi and Swamy 2005; Dobzinski et al. 2006; Nisan 2007; Dobzinski 2007]. A valuation $\mathbf{v}$ is subadditive if $\mathbf{v}(S \cup T) \leq \mathbf{v}(S) + \mathbf{v}(T)$, for all $S, T \subseteq J$. We show that for buyers with subadditive valuation functions, a random single price achieves revenue within a $2^{O(\sqrt{\log n \log \log n})}$ factor of the maximum social welfare.[3] We complement this result with a lower bound showing a sequence of subadditive (in fact, XOS) buyers for which the revenue of any single price is at most an $2^{\Omega(\log^{1/4} n)}$ fraction of the social optimum, thus showing that single prices cannot achieve a polylogarithmic ratio. Moreover, this lower bound holds even if the price is determined based on advance knowledge of the order and valuations of the buyers. The construction in this lower bound demonstrates a clear distinction in this setting between revenue maximization and social welfare maximization, for which [Dobzinski et al. 2006; Dobzinski 2007] show that a fixed price achieves a logarithmic approximation in the case of XOS [Dobzinski et al. 2006], and more generally subadditive [Dobzinski 2007], customers. We also show that even if we assume buyers arrive in a *random*

[1] Quasi-linear utilities means that buyers prefer the set maximizing the difference between its cost and its value. We assume that the sequence and valuations of buyers is determined in advance of any randomization made by the seller.

[2] A single-minded buyer is one who places some value v on a single set S or any superset of S, and value 0 on any set that does not contain S. A unit-demand buyer is one who has separate values v_j on each item j, and values any given set S at $\max_{j \in S} v_j$.

[3] The result given for the unlimited supply setting turns out to provide a useful structural characterization for proving the desired approximation for subadditive valuations in the limited supply case.

order, there exists a set of buyers for which a $2^{\Omega(\log^{1/4} n)}$ lower bound still holds. Note that our $2^{O(\sqrt{\log n \log\log n})}$ upper bound is the best approximation known for *any* item pricing scheme for subadditive buyers, even if assigning different prices to different items is allowed. We also show that for a special case we call *simple submodular valuations* (which generalizes unit-demand, additive, and submodular symmetric valuations [Lehmann et al. 2006]), a random single price does in fact achieve revenue within a logarithmic factor of the optimum social welfare.

Finally, we consider the multi-unit auctions setting [Dobzinski and Nisan 2007; Lehmann et al. 2006] where we have just multiple copies of a single item, but buyers have *arbitrarily complicated* valuation functions over the number of copies received. We show that under the assumption that the optimal allocation gives at most a $(1-\epsilon)$ fraction of the items to any one buyer, our single pricing scheme achieves a logarithmic approximation in this setting as well.

Related work: A similar result for the unlimited supply setting was later discovered in a different context by Briest et al. [2008] who study single price schemes in a network setting. In their setting, a buyer has certain subgraphs of the network it is interested in purchasing. A seller, who owns the network, first prices the edges and then the buyer purchases the cheapest subgraph it is interested in. They show that a single fixed price for all the edges guarantees the seller a revenue within logarithmic factor of the highest possible revenue.

In the context of designing computationally efficient mechanisms for social welfare maximization, work most related to ours is that of Dobzinski et al. [Dobzinski et al. 2006; Dobzinski 2007], who show that in the limited supply setting, a fixed price achieves a logarithmic approximation in the case of XOS or subadditive [Dobzinski et al. 2006; Dobzinski 2007] customers. In our work we analyze its power for maximizing revenue.

Discussion and Open Questions: Note that our lower bound for limited supply does not apply if one allows the seller to use different prices on different items. An interesting open question is whether an improved upper bound is possible using multiple prices, or on the other hand whether an alternative lower bound can be given for that case. In particular, it is an open question if the lower bound can be extended even to the case where the seller is allowed to use just *two* prices. A second open question is whether our lower bound (which uses XOS buyers) can be extended to the more restricted class of *submodular* buyers, or whether alternatively a polylog(n) upper bound can be obtained if buyers have submodular valuation functions. Finally, all our bounds are with respect to the social optimum; it would be interesting to show improved approximation (competitive ratio) guarantees with respect to the best fixed item pricing for the given sequence of bidders.

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