Solution to Exchanges 7.1 Puzzle: Combinatorial Auction Winner Determination

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This is a solution to the editor's puzzle from Issue 7.1 of SIGecom Exchanges. The puzzle is about solving an instance of the winner determination problem and providing a proof of optimality. The full puzzle [Conitzer] can be found online at `http://www.sigecom.org/exchanges/volume_7/1/puzzle.pdf`.

The puzzle asks us to determine the optimal allocation for a combinatorial auction with 5 items, *A, B, C, D, E,* and 12 (single-minded) bids. Let $j = 1, \ldots, 12$ stand for the submitted bids, and

$$x_j = \begin{cases} 1, & \text{if bid } j \text{ is accepted} \\ 0, & \text{if bid } j \text{ is rejected} \end{cases} \qquad j = 1, \ldots, 12$$

Then the standard winner determination problem can be written as the integer linear program:

$$\begin{array}{ll} \max & 5x_1+10x_2+24x_3+51x_4+13x_5+27x_6+43x_7+29x_8+25x_9+48x_{10}+14x_{11}+23x_{12} \\ & x_1+x_2+x_3+x_4 \leq 1 \\ & x_1+x_5+x_6+x_7+x_8 \leq 1 \\ (1) & x_2+x_3+x_4+x_5+x_6+x_9+x_{10} \leq 1 \\ & x_3+x_4+x_6+x_7+x_9+x_{10}+x_{11} \leq 1 \\ & x_4+x_7+x_8+x_{10}+x_{12} \leq 1 \\ & x_j \in \{0,1\} \quad \forall j \end{array}$$

where the inequality constraints indicate that each item can be sold to at most one bidder.

The linear programming relaxation of (1) has the nonintegral optimal solution (using an LP solver)

$$\tilde{x} = (0.5, 0, 0, 0.5, 0, 0, 0, 0.5, 0.5, 0, 0, 0).$$

We make use of the Chvatal-Gomory procedure (as described in [Papadimitriou]) to obtain an integral solution. Choosing multipliers $\mu = (0.5, 0.5, 0.5, 0.5, 0.5)$, i.e. multiplying each inequality constraint in (1) by 0.5 and summing them up yields

$$x_1 + x_2 + 1.5x_3 + 2x_4 + x_5 + 1.5x_6 + 1.5x_7 + x_8 + x_9 + 1.5x_{10} + 0.5x_{11} + 0.5x_{12} \leq 2.5$$

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Since the x_j are assumed to be integral and nonnegative, rounding down the coefficients gives rise to the valid inequality

$$x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \leq 2$$

Adding this inequality to the linear programming relaxation of (1) we solve

$$\begin{array}{llllllllllllll}
\max 5x_1 + 10x_2 + 24x_3 + 51x_4 + 13x_5 + 27x_6 + 43x_7 + 29x_8 + 25x_9 + 48x_{10} + 14x_{11} + 23x_{12} \\
x_1+ & x_2+ & x_3+ & x_4 & & & & & & & & & \leq 1 \\
x_1 & & & + & x_5+ & x_6+ & x_7+ & x_8 & & & & & \leq 1 \\
& x_2+ & x_3+ & x_4+ & x_5+ & x_6 & & + & x_9+ & x_{10} & & & \leq 1 \\
& & x_3+ & x_4 & + & x_6+ & x_7 & + & x_9+ & x_{10}+ & x_{11} & & \leq 1 \\
& & & x_4 & & + & x_7+ & x_8 & + & x_{10} & + & x_{12} & \leq 1 \\
x_1+ & x_2+ & x_3+ & 2x_4+ & x_5+ & x_6+ & x_7+ & x_8+ & x_9+ & x_{10} & & & \leq 2 \\
& & & & & & & & & & & x_j \geq & 0 \quad \forall j
\end{array} \tag{2}$$

Note that the constraint $x_j \leq 1$ is redundant in the relaxation of (1) since $x_j \geq 0$ and the inequality constraints already imply $x_j \leq 1$.

An LP solver computes an integral solution $x_8^*, x_9^* = 1, x_j^* = 0 \quad \forall j \notin \{8, 9\}$ for (2). To show that x^* is the optimal solution for (1) we only need to show that it is optimal for (2), because $x^* \in \{0,1\}^{12}$ then implies optimality for (1). The following theorem derived from the Complementary Slackness Theorem (see [Chvatal]) can be applied to show optimality of x^* for (2).

Theorem 0.1 (known). *Consider the LP*

$$\begin{aligned}
&\max \sum_{j=1}^{n} c_j x_j \\
&\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
&x_j \geq 0, \quad j = 1, \ldots, n
\end{aligned} \tag{3}$$

A feasible solution $x_1^*, x_2^*, \ldots, x_n^*$ of (3) is optimal if and only if there are numbers $y_1^*, y_2^*, \ldots, y_m^*$ such that

$$\begin{aligned}
\sum_{i=1}^{m} a_{ij} y_i^* = c_j \quad &\textit{whenever} \quad x_j^* > 0 \\
y_i^* = 0 \quad &\textit{whenever} \quad \sum_{j=1}^{n} a_{ij} x_j^* < b_i
\end{aligned} \tag{4}$$

and such that

$$\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \quad \forall j = 1, \ldots, n \tag{5}$$

$$y_i^* \geq 0 \quad \forall i = 1, \ldots, m \tag{6}$$

In problem (2) we have $n = 12, m = 6$ and it is easy to verify that $y^* = (0, 3, 8, 14, 23, 3)$ satisfies the conditions of the theorem and hence proves optimality of $x^* = (0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0)$. So the winning bids are (8) \$29 for $\{B, E\}$ and (9) \$25 for $\{C, D\}$, thus in the optimal solution all items except A are sold at a total revenue of \$54.

REFERENCES

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