Competitive Equilibria in Matching Markets with Budgets

NING CHEN
Nanyang Technological University, Singapore
and
XIAOTIE DENG
City University of Hong Kong, Hong Kong
and
ARPITA GHOSH
Yahoo! Research, Santa Clara, CA, USA

1. INTRODUCTION

Consider a market with n unit demand buyers and m sellers, each selling one unit of an indivisible good. The buyers specify their preferences over items via utility functions $u_{ij}(p_j)$, which is the utility of buyer i for item j when its price is p_j. So far, this is the classic Shapley-Shubik assignment model [Shapley and Shubik 1971] which captures a variety of matching markets including housing markets and ad auctions [Edelman et al. 2007], except for the extension to general utility functions instead of the quasi-linear utilities in the original model. Shapley and Shubik show that a *competitive equilibrium* always exists in their model, and later work [Crawford and Knoer 1981, Quinnzi 1984, Gale 1984] shows that a competitive equilibrium must also exist for the model with general utility functions $u_{ij}(\cdot)$, provided these $u_{ij}(\cdot)$ are strictly decreasing and continuous everywhere.

Now suppose we extend the assignment model with an extra *budget constraint*: a buyer i can specify a maximum price b_{ij} that he is able to pay for item j, above which he cannot afford the item. That is, the utility function $u_{ij}(\cdot)$ can now (possibly) have a *discontinuity* at $p_j = b_{ij}$. Budgets are a very real constraint in many marketplaces such as advertising markets, and have led to a spate of recent work on auction design, where the addition of the budget constraint, while seemingly innocuous, introduces fundamental new challenges to the problem [Dobzinski et al. 2008]. As we will see, the same happens in the Shapley-Shubik assignment model — the discontinuity introduced by the budget constraint fundamentally changes the properties of competitive equilibria, which no longer even always exist. A natural question, then, is the following: Given a marketplace where buyers have such general utility functions $u_{ij}(\cdot)$ with budget constraints, is it possible to determine whether a competitive equilibrium exists, and if yes, compute one, for instance a buyer-optimal one, efficiently?

Authors' addresses: ningc@ntu.edu.sg, csdeng@cityu.edu.hk, arpita@yahoo-inc.com

2. CONTRIBUTIONS

We first make a connection between competitive equilibrium in the assignment model with budgets, and *strong stability.* Next, we give a strongly polynomial time algorithm for deciding existence of, and computing a minimum competitive equilibrium for a general class of utility functions in the assignment model with budgets. A complete description can be found in the full version of our paper [Chen et al. 2010].

2.1 Stability

Shapley and Shubik [Shapley and Shubik 1971] make a connection between competitive equilibria and stability in the assignment model, showing that competitive equilibrium corresponds precisely to stable matchings — a matching where no unmatched buyer-seller pair can mutually benefit by trading with each other instead of their current partners. This connection becomes somewhat more subtle when utility functions are allowed to have budget-induced discontinuities, analogous to the introduction of ties into preference lists [Irving 1994] in the Gale-Shapley marriage model [Gale and Shapley 1962]: There are now two different notions of stability, *weak* and *strong* stability, that no longer coincide; weakly stable matchings always exist but a strongly stable matching need not.

Denote an allocation-price pair by $(\mathbf{x}, \mathbf{p})$, where the allocation x_i is the item that buyer i wins, and p_j is the price charged for item j. A *weakly stable* matching in the assignment model is a tuple $(\mathbf{x}, \mathbf{p})$ where there is no unmatched buyer-seller pair that can both strictly benefit by trading with each other (a seller's payoff is the payment he receives for his item). A *strongly stable* matching is one where there is no unmatched buyer-seller pair where one party strictly benefits and the other weakly benefits from the deviation (for example, if buyer i strictly prefers to buy seller j's item at its current price, but not at any higher price).

When the utility functions $u_{ij}(\cdot)$ are strictly decreasing and continuous everywhere as in [Crawford and Knoer 1981, Demange and Gale 1985], the conditions for weak and strong stability turn out to be identical: if i strictly prefers j, continuity ensures that it is always possible for i and j to deviate in such a way that both i and j strictly benefit from the deviation. So weakly and strongly stable matchings are identical, and there is a unique notion of stability in the original matching model; this notion is exactly the same as a competitive equilibrium.

However, with the budget constraint, this is no longer the case: suppose there is a pair (i, j) such that $u_{ij}(p_j) > u_{ix_i}(p_{x_i}) \geq 0$. Depending on whether $p_j < b_{ij}$ or $p_j = b_{ij}$, there may or may not exist a strictly profitable deviation for *both* i and j: in the first case, there exists a $p'_j > p_j$ with $u_{ij}(p'_j) > u_{ix_i}(p_{x_i})$; but in the second case, there is no $p'_j > p_j$ for i to continue to prefer j over x_i, i.e., there is no strictly profitable deviation for j. That is, with the addition of the budget constraint, the two notions of "weak" and "strong" stability are no longer equivalent: the non-equivalence is precisely because a buyer's utility can go from strictly positive to strictly negative without passing through 0 at the point of discontinuity at b_{ij}.

Since there are two distinct notions of stability in the matching model with budgets, at most one of these can be the same as a competitive equilibrium. We prove that the solution concept of a competitive equilibrium coincides exactly with that of strong stability; a weakly stable matching does not possess the envy-freeness property of a competitive equilibrium (the problem of computing weakly stable matchings in the assignment model with quasi-linear utilities up to a budget constraint is introduced and solved in [Aggarwal et al. 2009]).

THEOREM 2.1. *Suppose that the utility functions* $u_{ij}(\cdot)$ *are strictly decreasing on domain* $[0, b_{ij}]$*, where* b_{ij} *is buyer* i*'s budget for an item* j*. Then,* $(\mathbf{x}, \mathbf{p})$ *is a competitive equilibrium if and only if it is strongly stable.*

2.2 Computation

Our main result answers the question of computing competitive equilibria, if they exist, for a general class of utility functions $u_{ij}(\cdot)$ which are continuous and strictly decreasing on $[0, b_{ij}]$ as in [Crawford and Knoer 1981, Demange and Gale 1985], and satisfy an additional mutual *consistency* condition that allows increasing prices in a way that guarantees strongly polynomial runtime. This class of utility functions, which we will call *consistent* utility functions, is quite general and allows modeling a fairly large class of marketplaces:

—Marketplaces with buyers who have quasi-linear utilities and budgets, $u_{ij}(p_j) = v_{ij} - p_j$ for $p_j \leq b_{ij}$, and negative utility for $p_j > b_{ij}$, where v_{ij} is the value of buyer i for item j and b_{ij} is the corresponding budget.

—Marketplaces with return-on-investment (ROI) based buyers with budget constraints, i.e., buyers who want to maximize the ratio $u_{ij}(p_j) = v_{ij}/p_j$ subject to a limit on their payment. Our model also captures buyers who know only the relative values v_{ij}/v_{i1} of items for $j = 2, \ldots, m$ (v_{i1} is i's value for the first item), and may not know exactly the magnitudes of their values v_{ij}.

—Marketplaces where buyers who can only *rank* items in order of preference, and have budget constraints for each item. For example, a buyer who prefers item j_1 over all other items as long as its price is less than or equal to b_{ij_1}, else prefers item j_2 as long as price is less than or equal to b_{ij_2}, and so on; a simple special case is a buyer who has a fixed preference ranking over items and a single budget constraint.

We give an algorithm that provides a constructive proof of the following result (see Definition 4.1 [Chen et al. 2010] for a precise definition of the minimum equilibrium):

THEOREM 2.2. *Suppose we are given an instance of the assignment model with consistent utility functions* u_{ij}*. Then, if a competitive equilibrium exists, a minimum competitive equilibrium exists as well; further, the problem of deciding whether or not an equilibrium exists, and computing a minimum one, can be solved in strongly polynomial time.*

The algorithm starts with the zero price vector $\mathbf{p} = 0$, and repeatedly constructs bipartite *dynamic demand graphs* $G(\mathbf{p})$ based on the demand sets — the set of items with maximal, positive, utility at the current prices $\mathbf{p}$. It then identifies the set of "over-demanded" items in this demand graph, which is captured by the *critical set* A of buyers and its neighborhood $N(A)$ in $G(\mathbf{p})$: not all buyers in A can be matched to distinct items in $N(A)$. Therefore, there can be no equilibrium at price

p, and the prices of items in $N(A)$ need to be raised until these items are no longer over-demanded. The algorithm recursively increases the prices of over-demanded items, and once all critical sets have been eliminated, checks whether there is a matching satisfying the conditions for a competitive equilibrium — all buyers are assigned an item in their demand set, and every item with price greater than zero is assigned to a buyer.

There are two key challenges to developing a strongly polynomial time algorithm that returns a minimum competitive equilibrium in our model. The first comes from the fact that we allow rather general utility functions: the algorithm needs to increase prices of over-demanded items in each step to the maximum extent possible to ensure a fast runtime, while making sure that prices do not overshoot any equilibrium price vectors. However, to ensure that the algorithm does successfully find a minimum equilibrium or else correctly reports that no equilibrium exists, it is crucial that *only* the prices of items in the neighborhood of the critical set are increased, at every vector of prices through the course of the algorithm. Due to the generality of the utility functions we consider, it is possible that the demand set of a buyer $i \in A$ changes "non-monotonically" if the prices of items in $N(A)$ are not increased carefully, in the sense that items can drop out and then return again to i's neighborhood, potentially leading to an exponential runtime. The consistency property of the utility functions allows us to develop a subroutine that increases the prices of over-demanded items in such a way that an item j vanishes from a buyer i's demand graph *if and only if* $u_{ij}(p_j) \leq 0$, i.e., only at the price at which i can no longer obtain positive utility from j. An edge (i, j) that is removed from the demand graph never appears again through the remainder of the algorithm; this is crucial to the strongly polynomial runtime.

The second challenge is to deal with the discontinuity introduced by the budget constraint: For any edge (i, j) in the demand graph, while buyer i's utility from buying an item passes *continuously* through 0 at the value constraint v_{ij} (where $u_{ij}(v_{ij}) = 0$, i.e., i is indifferent between buying and not buying the item), there is a discontinuity at the budget b_{ij} where i is *not indifferent* between these two actions. While a buyer obtains negative utility at any price higher than either v_{ij} or b_{ij}, our algorithm needs to account for a change in edge structure in the demand graph differently depending on which of these two thresholds is reached: If an edge is dropped at $p_j = v_{ij}$, an equilibrium can exist at this price, whereas at price $p_j = b_{ij}$ buyer i still strictly prefers to buy the item, and so j must be priced strictly higher in any equilibrium. The algorithm accounts for the discontinuity introduced by the budget using a careful marking process that tags such items whose prices need to be increased later to ensure envy-freeness. The set of marked items in the final output of the algorithm also has a nice property relating weakly and strongly stable matchings (Proposition D.1 [Chen et al. 2010]).

3. CONCLUSION

Our original motivation for studying the assignment model with general utility functions comes from matching markets such as the online and TV advertising markets, where bidders do not always fit the standard model of quasi-linear utility optimizers. One obvious example of this is advertisers with budget constraints;

another is advertisers who may not be able to accurately estimate their values for items, but only be able to specify a preference ordering amongst items along with a maximum payment limit for an item (because the precise value of placing an ad depends on factors such as conversion rates which are difficult to estimate, while determining the relative ordering amongst different placement options is much easier). Such bidders cannot specify their values for each item in the quasi-linear utility model either. A third kind of bidders that do not fit the standard model are advertisers who want to optimize return-on-investment (ROI), which is value divided by price rather than value minus price; these advertisers might also be able to better estimate the ratios of values rather than the values themselves. An algorithm that computes a fair and efficient assignment of items with such general inputs can significantly improve the advertising marketplace by making it much easier for advertisers to participate and bid in the market. Of course, clearing a real advertising marketplace would require solving a model with multiple units of supply and demand for each seller and buyer — we take the first steps towards this rather difficult problem by solving the market clearing problem for the unit demand case.

REFERENCES

G. Aggarwal, S. Muthukrishnan, D. Pal, M. Pal, *General Auction Mechanism for Search Advertising*, WWW 2009, 241-250.

N. Chen, X. Deng, A. Ghosh, *Competitive Equilibria in Matching Markets with Budgets*, `http://arxiv.org/PS_cache/arxiv/pdf/1004/1004.2565v2.pdf`.

V. Crawford, E. Knoer, *Job Matching with Heterogeneous Firms and Workers*, Econometrica, V.49(2), 437-450, 1981.

G. Demange, D. Gale, *The Strategy of Two-Sided Matching Markets*, Econometrica, V.53, 873-888, 1985.

S. Dobzinski, R. Lavi, N. Nisan, *Multi-Unit Auctions with Budget Limits*, FOCS 2008, 260-269.

B. Edelman, M. Ostrovsky, M. Schwarz, *Internet Advertising and the Generalized Second-Price Auction*, American Economic Review, 97(1), 242-259, 2007.

D. Gale, *Equilibrium in a Discrete Exchange Economy with Money*, International Journal of Game Theory, V.13, 61-64, 1984.

D. Gale, L. Shapley, *College Admissions and the Stability of Marriage*, American Mathematical Monthly, V.69, 9-15, 1962.

R. W. Irving, *Stable Marriage and Indifference*, Discrete Applied Mathematics, V.48, 261-272, 1994.

M. Quinzii, *Core and Competitive Equilibria with Indivisibilities*, International Journal of Game Theory, V.13, 41-60, 1984.

L. Shapley, M. Shubik, *The Assignment Game I: The Core*, International Journal of Game Theory, V.1(1), 111-130, 1971.